A generalization of
Vassiliev's h-principle

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Lukáš Vokřínek

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Abstract

This thesis consists of two parts which share only a slight overlap.

The first part is concerned with the study of ideals in the ring $C^\infty(M, \mathbb{R})$ of smooth functions on a compact smooth manifold M or more generally submodules of a finitely generated $C^\infty(M, \mathbb{R})$-module $\mathcal{V}$. We define a topology on the space $\mathrm{M}_d(\mathcal{V})$ of all submodules of $\mathcal{V}$ of a fixed finite codimension d. Its main property is that it is compact Hausdorff and, when $\mathcal{V} = C^\infty(M, \mathbb{R})$, it contains as a subspace the configuration space of d distinct unordered points in M and therefore gives a "compactification" of this configuration space. We present a concrete description of this space for low codimensions.

The main focus is then put on the second part which is concerned with a generalization of Vassiliev's h-principle. This principle in its simplest form asserts that the jet prolongation map $j^r : C^\infty(M, V) \to \Gamma(J^r(M, V))$, defined on the space of smooth maps from a compact manifold M to a Euclidean space V and with target the space of smooth sections of the jet bundle $J^r(M, V)$, is a cohomology isomorphism when restricted to certain "nonsingular" subsets (these are defined in terms of a certain subset $R \subseteq J^r(M, V)$). Our generalization then puts this theorem in a more general setting of topological $C^\infty(M, \mathbb{R})$-modules. As a reward we get a strengthening of this result asserting that all the homotopy fibres have zero homology.

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CHAPTER 0

Introduction

The history of h-principles (homotopy principles) goes back to Smale who classified in [**Sma**] immersions of spheres into a Euclidean space up to regular homotopy and to Hirsch who generalized Smale's work to immersions $M \to N$ between any manifolds (in [**Hir1**] for $\dim M < \dim N$ and in [**Hir2**] for $\dim M = \dim N$ with M open). More elaborated h-principles can be stated as follows. Let $R \subseteq J^r(M, N)$ (corresponding to jets of immersions in the preceding) and let us consider the set

$$\Gamma R := \{s \in \Gamma(J^r(M, N)) \mid \mathrm{im}\, s \subseteq R\}$$

of those sections of the jet bundle $J^r(M, N) \to M$ whose image lies in R. The subset

$$\Gamma_{\mathrm{hol}} R := \{j^r f \in \Gamma R \mid f \in C^\infty(M, N)\}$$

of holonomic sections can be clearly identified with

$$\{f \in C^\infty(M, N) \mid j^r_x f \in R \; \forall x \in M\} \subseteq C^\infty(M, N)$$

and this identification is in fact a homeomorphism. A (parametric) h-principle for R generally asserts that the inclusion $\Gamma_{\mathrm{hol}} R \hookrightarrow \Gamma R$ is a weak homotopy equivalence. The statements about the set of immersions up to regular homotopy are then translated into ones about π_0. In [**Gro**] Gromov proved that the h-principle holds for any open $\mathrm{Diff}\, M$-invariant R provided that M is open.

For the case of compact M the situation is more complicated and in fact the h-principle for immersions between manifolds of the same dimension does not hold. A partial result for maps $M \to V$ into a Euclidean space V is given in [**Vas**]: Vassiliev proves that if R is open and if its complement is a semialgebraic $\mathrm{Diff}\, M$-invariant subset of codimension at least $\dim M + 2$ then the inclusion $\Gamma_{\mathrm{hol}} R \to \Gamma R$ is a cohomology isomorphism. Moreover he constructs a spectral sequence converging to the cohomology of these spaces.

The purpose of this thesis is to generalize this theorem in few ways. Our proof is based upon interpolation theory and transversality theory. Both these work in a more general setting than $C^\infty(M, V)$ and allow us to construct Vassiliev's spectral sequence for the homotopy fibres of the inclusion $\Gamma_{\mathrm{hol}} R \to \Gamma R$ proving that they are acyclic (have zero homology). This is again under the assumption that the codimension of the complement of R in $J^r(M, V)$ is at least $\dim M + 2$.

Our setting is that of topologically finitely generated affine $C^\infty(M, \mathbb{R})$-modules (topological quotients of free $C^\infty(M, \mathbb{R})$-module of finite rank). A "representation" of such a module $\mathcal{V}$ on an affine bundle $E \to M$ is a special map $\varphi : \mathcal{V} \to \Gamma E$ which enables us to determine whether an element $v \in \mathcal{V}$ "lies in" an open subset $R \subseteq E$: let us call v regular if $\varphi(v) \subseteq R$. Then we can consider the subset of all regular v and ask

what its homotopy type is (or homology groups in our case). The space $\Gamma_{\mathrm{hol}}R$ is the space of regular elements for

$$j^r : \mathcal{W}_{\mathrm{hol}} = C^\infty(M, V) \to \Gamma(J^r(M, V))$$

and ΓR the space of regular elements for

$$\mathrm{id} : \mathcal{W} = \Gamma(J^r(M, V)) \to \Gamma(J^r(M, V))$$

Interesting examples of topologically finitely generated affine $C^\infty(M, \mathbb{R})$-modules arise as affine submodules of $\mathcal{W}_{\mathrm{hol}}$ of functions whose r-jets along a submanifold (for example along the boundary or at a point) are fixed and the corresponding affine submodules of $\mathcal{W}$.

Returning to the general case we denote for simplicity by A the complement of R in E, by φ_x the composition $\mathcal{V} \xrightarrow{\varphi} \Gamma E \xrightarrow{ev_x} E_x$ for $x \in M$ and by $\mathcal{V}_A$ the set of regular elements. Our main theorem (Theorem 6.1 in the main text) can be stated as follows:

THEOREM. *Let M be a compact smooth manifold, let $\alpha : \mathcal{U} \to \mathcal{V}$ be an affine $C^\infty(M, \mathbb{R})$-homomorphism between two topologically finitely generated affine $C^\infty(M, \mathbb{R})$-modules, let $\varphi : \mathcal{V} \to \Gamma E$ be a "representation" on an affine bundle $E \to M$ and $A \subseteq E$ a manifold stratified subset of codimension at least $\dim M + 2$ such that outside the set[1]*

$$\hat{M} = \{x \in M \mid (\varphi\alpha)_x \text{ is surjective}\}$$

we have $\mathrm{im}\,\varphi_x \cap A = \emptyset$. Then each homotopy fibre $\mathrm{hofib}_v\alpha_A$ of the restriction $\alpha_A : \mathcal{U}_A \to \mathcal{V}_A$ of α to the sets of regular elemets is acyclic, i.e. $\tilde{H}_*(\mathrm{hofib}_v\alpha_A) = 0$.

Now for the actual contents of the thesis. The first chapter is concerned with the interpolation on smooth manifolds. All that is required for further chapters is the first section which guarantees an existence of a finite dimensional linear subspace in any finitely generated $C^\infty(M, \mathbb{R})$-module $\mathcal{V}$ that is transverse to all submodules of a fixed finite codimension. In the case of the ring $C^\infty(M, \mathbb{R})$ itself we have special submodules/ideals for points $x_1, \ldots, x_n \in M$:

$$\{f : M \to \mathbb{R} \mid j^r_{x_1}f = \cdots = j^r_{x_n}f = 0\}$$

For a subspace $D \subseteq C^\infty(M, \mathbb{R})$ to be transverse to this ideal is equivalent to the multi-jet evaluation map

$$(j^r_{x_1}, \ldots, j^r_{x_n}) : C^\infty(M, \mathbb{R}) \to J^r_{x_1}(M, \mathbb{R}) \times \cdots \times J^r_{x_n}(M, \mathbb{R})$$

being surjective when restricted to D, i.e. given any collection of r-jets at points $x_1, \ldots, x_n$ there is a function lying in D that realizes them. This is the way the interpolation property is used in the main proof.

The remainder of Chapter 1 is devoted to studying the space of submodules of $\mathcal{V}$ of a fixed finite codimension. Generalizing Glaeser's article [**Gla**] it is given a canonical topology making it into a compact Hausdorff space. We present a few examples showing what this topology looks like. We also include a criterion for an $C^\infty(M, \mathbb{R})$-module to be topologically finitely generated justifying the above example of submodules with fixed r-jets along a submanifold.

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[1]This (a bit technical) condition is justified by the example of maps with a fixed r-jet along a submanifold where one could allow only jets in $R = E - A$.

The second and the third chapters are preparing the ground for the construction of the main spectral sequence.

In the fourth chapter, assuming that the codimension of A in E is at least $\dim M + 1$, we derive the main spectral sequence for the homology of $\mathcal{V}_A$ and show that it converges if this codimension is at least $\dim M + 2$. As promised we use the interpolation theory of (the first section of) Chapter 1 and the transversality theory of Chapter 3.

In Chapter 5 we construct the homotopy fibre of the map α_A as the space $\mathcal{W}_{A \times I}$ of regular elements in some topologically finitely generated affine $C^\infty(M \times I, \mathbb{R})$-module $\mathcal{W}$ equipped with a representation $\mathcal{W} \to \Gamma(E \times I)$ on the bundle $E \times I \to M \times I$. This allows us to apply our spectral sequence on it and finally in Chapter 6 prove our main result.

CHAPTER 1

Interpolation on smooth manifolds

Let us first state clearly that this chapter is based on the article [**Gla**] of Glaeser. In a sense it is just a (nontrivial) generalization of the ideas of this article from parallelepiped in $\mathbb{R}^m$ to general compact manifolds. The main structure of the proof of the compactness of the space of ideals (or more generally submodules) remains the same although at some point one has to come up with a new approach as Glaeser's proof is very "affine". In the first section we gather well-known facts about the ideals in the algebra of smooth functions. The results in subsequent sections are original.

In this chapter let M be a compact smooth manifold, $\mathcal{R} = C^\infty(M, \mathbb{R})$ the ring of smooth functions on M. We will be considering finitely generated $\mathcal{R}$-modules $\mathcal{V}$ and their submodules of a fixed finite codimension d over $\mathbb{R}$. Our ultimate goal is to endow the set M_d of all such submodules with a topology. It will be compact Hausdorff. Together with M_d we will also topologize

$$\mathrm{E}_d = \{(B, w) \mid B \in \mathrm{M}_d, \ w \in \mathcal{V}/B\}$$

in such a way that the canonical projection map $\mathrm{E}_d \to \mathrm{M}_d$ will become a vector bundle. If $\mathcal{V}$ is a topological $\mathcal{R}$-module and it is topologically finitely generated then the map

$$\mathcal{V} \times \mathrm{M}_d \longrightarrow \mathrm{E}_d$$

sending (v, B) to $(B, v + B)$ is a continuous homomorphism of vector bundles. This is certainly a property one would require from any such topology.

To demonstrate other properties let us now specialize to the case $\mathcal{V} = \mathcal{R}$ so that M_d is the set of all ideals of $\mathcal{R}$ of codimension d. For any set $Y \subseteq M$ consisting of d points one has an ideal

$$\mathfrak{m}_Y := \{f \in \mathcal{R} \mid f(y) = 0 \text{ for all } y \in Y\}$$

and easily $\mathfrak{m}_Y \in \mathrm{M}_d$. In this way one can embed into M_d the configuration space $M^{[d]}$ of d distinct unordered points in M

$$M^{[d]} \hookrightarrow \mathrm{M}_d$$

For our topology on M_d this inclusion map is a topological embedding and therefore one can think of M_d as a "compactification" of $M^{[d]}$.

There is an inverse procedure of associating to each $I \in \mathrm{M}_d$ an unordered d-tuple of points in M. However for ideals not of the form $\mathfrak{m}_Y$ these points do not have to be distinct and therefore this procedure yields a map

$$\mathrm{wsp} : \mathrm{M}_d \to M^d/\Sigma_d$$

We call wsp(B) the weighted spectrum of B. For any reasonable topology on M_d this map should be continuous as well. This is indeed the case for our topology.

Let us try to indicate now how we construct the topology on M_d. If $\mathcal{V}$ was finite dimensional we could simply think of M_d as a subset of the Grassmannian manifold $G_d(\mathcal{V})$ of linear subspaces of $\mathcal{V}$ of codimension d. This is of course almost never the case. On the other hand suppose that there is a finite dimensional linear subspace $D \subseteq \mathcal{V}$ that is transverse to each $B \in \mathrm{M}_d$. Then the intersection with D produces a map
$$\mathrm{M}_d \longrightarrow G_d(D)$$
from M_d to the Grassmannian manifold of codimension d linear subspaces of D. If this map is moreover injective one can give M_d the subspace topology. The key step is therefore to show that there is a transversal D for which the map $\mathrm{M}_d \longrightarrow G_d(D)$ is injective. This is the first section and it is rather elementary. It starts by describing the structure of submodules of a given $\mathcal{R}$-module $\mathcal{V}$ so that one is able to understand this transversality condition. At the end of the section we produce a transversal.

In the second section we derive an "interpolation formula". The idea is that if D is transverse to some submodule $B \in \mathrm{M}_d$ then for each $v \in \mathcal{V}$ there is some $d \in D$ such that $v = d$ modulo B. Therefore one can "interpolate" elements of $\mathcal{V}$ "at B" by elements of D. Moreover if we assume that the dimension of D is precisely d there is exactly one such interpolation. Intuitively D should then be also transverse to all submodules that are close to B and one should get a continuous interpolation map
$$\mathcal{V} \times \mathrm{Nbhd}(B) \to D$$
A problem is that M_d does not posses any topology so far and so we cannot talk about a neighbourhood of B. However we have a topology on M^d/Σ_d. Using locally the affine structure of M we construct a continuous interpolation map[1]
$$\mathcal{V} \times \mathrm{Nbhd}(Y) \to D$$
where $Y \in M^d/\Sigma_d$ is the weighted spectrum of B. It serves as a tool both in the proof of the compactness of M_d and in the proofs of its main properties.

The third section is dedicated to describing the topology of $\mathrm{M}_d(\mathcal{R})$ in some special cases (codimensions 1, 2 and 3 and the case of one-dimensional manifolds). Along with these examples two general theorems are proved, the first of which states that the inclusion of the configuration space into M_d is a smooth embedding (as was already mentioned above). The second theorem is concerned with the set of all ideals I with a fixed isomorphism type of the quotient algebra $\mathcal{R}/I$. These ideals possess a structure of a smooth manifold (as was proved in [**Alo**]) and the content of our second theorem is that they form an immersed submanifold in M_d. As a set M_d is then a disjoint union of these submanifolds.

One of the important notions that turn up in the course of this chapter is the following. An $\mathcal{R}$-module $\mathcal{V}$ is topologically finitely generated if $\mathcal{V}$ is a topological quotient of a free $\mathcal{R}$-module of finite rank (with its canonical topology). In the fourth section we give a simple criterion for an $\mathcal{R}$-module to be topologically finitely generated. It

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[1]One has to specify what an interpolation property with respect to elements of M^d/Σ_d means. It forces taking a slightly bigger D and as a result there is no preferred/unique interpolation map.

suffices for it to be locally topologically finitely generated. As an application we give some examples of topologically finitely generated modules.

Finally in the last section we briefly discuss the results for non-compact manifolds. This is still a work in progress and as a result almost no proofs are included.

1.1. The structure of submodules of $\mathcal{V}$

For the first few results we do not need to restrict to finitely generated modules, hence we assume now that $\mathcal{V}$ is any $\mathcal{R}$-module. We investigate the structure of submodules of $\mathcal{V}$, first looking at the special case of ideals of the ring $\mathcal{R}$ itself.

LEMMA 1.1. *The maximal ideals in $\mathcal{R}$ are identified with points in M. For a point $x \in M$ the corresponding ideal is*

$$\mathfrak{m}_x := \{f : M \to \mathbb{R} \mid f(x) = 0\}$$

PROOF. To prove that $\mathfrak{m}_x$ is maximal it is enough to observe that it is the kernel of the evaluation homomorphism $ev_x : \mathcal{R} \to \mathbb{R}$ and that $\mathbb{R}$ is a field.

On the other hand suppose that I is a proper ideal and we will prove that there is $x \in M$ such that $I \subseteq \mathfrak{m}_x$. Assume on the contrary that no such point exists. Then for every point x there is a function $f_x \in I$ such that $f_x \neq 0$ on a neighbourhood U_x of x. Replacing f_x by f_x^2 if needed we can assume that each f_x is nonnegative. By compactness there is a finite set $\{x_1, \ldots, x_k\} \subseteq M$ such that $U_{x_1}, \ldots, U_{x_k}$ cover M. Therefore $f = f_{x_1} + \cdots + f_{x_k} \in I$ is nonzero on M and consequently a unit in $\mathcal{R}$, a contradiction to the properness of I. $\square$

For any ideal $I \subseteq \mathcal{R}$ the *spectrum* of I is defined to be the closed subset

$$\operatorname{sp}(I) := \bigcap_{f \in I} f^{-1}(0)$$

The previous lemma then says that it can also be described as the subset of those points $x \in M$ for which $I \subseteq \mathfrak{m}_x$. For any closed subset $A \subseteq M$ we denote by $\mathfrak{n}_A$ the ideal of functions which vanish on a neighborhood of A. The quotient $\mathcal{R}/\mathfrak{n}_A$ is then the ring of germs at A of smooth functions on M.

LEMMA 1.2. *Let $I \subseteq \mathcal{R}$ be an ideal and $A \subseteq M$ a closed subset. Then*

$$\operatorname{sp}(I) \subseteq A \quad \Leftrightarrow \quad \mathfrak{n}_A \subseteq I$$

REMARK. If we define $\mathfrak{m}_A$ to be the ideal of functions vanishing on A then also $A \subseteq \operatorname{sp}(I)$ if and only if $I \subseteq \mathfrak{m}_A$. Both these statements can be phrased as adjointness of the respective functors.

PROOF. Let $f \in \mathfrak{n}_A$ be any function $M \to \mathbb{R}$ vanishing on a neighborhood of A. We set $C = \operatorname{supp}(f)$ and we have

$$C \subseteq M - A$$

We can cover C by open subsets U_i for which there is $g_i \in I$ that is positive on U_i. By the means of a partition of unity we can glue them to get a function $g \in I$ which is positive on a neighborhood of C. Then $f = \frac{f}{g}g$ expresses f as an element of I. $\square$

LEMMA 1.3. *Let A and A' be closed subsets of M. Then the following holds*

(i) $\mathfrak{n}_A \cap \mathfrak{n}_{A'} = \mathfrak{n}_{A \cup A'}$
(ii) $\mathfrak{n}_A + \mathfrak{n}_{A'} = \mathfrak{n}_{A \cap A'}$

PROOF. The part (i) is obvious and so is the inclusion $\subseteq$ in the part (ii). Hence let $f \in \mathfrak{n}_{A \cap A'}$, $\hat{A} = A \cap \mathrm{supp}(f)$ and $\hat{A}' = A' \cap \mathrm{supp}(f)$. As $\hat{A} \cap \hat{A}' = \emptyset$ we can find a function $\lambda \in \mathcal{R}$ such that $\lambda = 0$ on a neighborhood of $\hat{A}$ and $\lambda = 1$ on a neighborhood of $\hat{A}'$. Then clearly $\lambda f \in \mathfrak{n}_A$ and $(1 - \lambda)f \in \mathfrak{n}_{A'}$ and so

$$f = \lambda f + (1 - \lambda)f \in \mathfrak{n}_A + \mathfrak{n}_{A'}$$

$\square$

LEMMA 1.4. *Let I and I' be ideals with disjoint spectra and $B \subseteq \mathcal{V}$ any submodule. Then*

(i) $II' = I \cap I'$
(ii) $II'\mathcal{V} = I\mathcal{V} \cap I'\mathcal{V}$
(iii) $B + II'\mathcal{V} = (B + I\mathcal{V}) \cap (B + I'\mathcal{V})$

PROOF. Obviously (iii) is the most general case. We give here only the proof of (ii) for the simplicity sake. Let us denote the spectra of I and I' by A and A' respectively. The inclusion $\subseteq$ is obvious and

$$I\mathcal{V} \cap I'\mathcal{V} = (\mathfrak{n}_A + \mathfrak{n}_{A'})(I\mathcal{V} \cap I'\mathcal{V}) = \mathfrak{n}_A(I\mathcal{V} \cap I'\mathcal{V}) + \mathfrak{n}_{A'}(I\mathcal{V} \cap I'\mathcal{V})$$
$$\subseteq \mathfrak{n}_A I'\mathcal{V} + \mathfrak{n}_{A'} I\mathcal{V} \subseteq II'\mathcal{V}$$

(the first equality follows from Lemma 1.3 and the last inclusion from Lemma 1.2). The proof for $B \neq 0$ follows the same idea. $\square$

Let $B \subseteq \mathcal{V}$ be any submodule. One has an ideal $(B : \mathcal{V})$ of $\mathcal{R}$ defined by

$$(B : \mathcal{V}) := \{f \in \mathcal{R} \mid f\mathcal{V} \subseteq B\}$$

Alternatively, it is the kernel of the action map

$$\mathcal{R} \to \mathsf{End}(\mathcal{V}/B)$$

We define a *spectrum* of B (inside $\mathcal{V}$) to be the spectrum of the ideal $(B : \mathcal{V})$ and denote it by $\mathrm{sp}_{\mathcal{V}}(B)$ or simply by $\mathrm{sp}(B)$.

LEMMA 1.5. *Let $B \subseteq \mathcal{V}$ be a submodule and $A \subseteq M$ a closed subset. Then*

$$\mathrm{sp}(B) \subseteq A \quad \Leftrightarrow \quad \mathfrak{n}_A \mathcal{V} \subseteq B$$

PROOF. This is clear as $\mathfrak{n}_A \mathcal{V} \subseteq B$ if and only if $\mathfrak{n}_A \subseteq (B : \mathcal{V})$ if and only if $\mathrm{sp}(B) = \mathrm{sp}(B : \mathcal{V}) \subseteq A$ according to Lemma 1.2. $\square$

Our next goal is to decompose and thus simplify the submodules of $\mathcal{V}$. First we need yet another lemma.

LEMMA 1.6. *Let B and B' be submodules of $\mathcal{V}$. Then the following holds.*

(i) *If $B \subseteq B'$ then $\mathrm{sp}(B') \subseteq \mathrm{sp}(B)$*

(ii) $\mathrm{sp}(B \cap B') = \mathrm{sp}(B) \cup \mathrm{sp}(B')$

(iii) $\mathrm{sp}(B + B') \subseteq \mathrm{sp}(B) \cap \mathrm{sp}(B')$

PROOF. To prove (i) one observes that $(B : \mathcal{V}) \subseteq (B' : \mathcal{V})$ and so

$$\mathrm{sp}(B') = \mathrm{sp}(B' : \mathcal{V}) \subseteq \mathrm{sp}(B : \mathcal{V}) = \mathrm{sp}(B)$$

For the inclusion $\subseteq$ of (ii) let us denote $A = \mathrm{sp}(B)$ and $A' = \mathrm{sp}(B')$ so that (using Lemma 1.5 in both directions)

$$\mathfrak{n}_{A \cup A'}\mathcal{V} \subseteq \mathfrak{n}_A \mathcal{V} \cap \mathfrak{n}_{A'}\mathcal{V} \subseteq B \cap B' \quad \Rightarrow \quad \mathrm{sp}(B \cap B') \subseteq A \cup A'$$

The remaining claims are trivial consequences of (i). $\qquad\square$

COROLLARY 1.7. *If $B_1, \ldots, B_n$ are submodules of $\mathcal{V}$ with pairwise disjoint spectra, then they are in general position, i.e.*

$$B_i + (B_1 \cap \cdots \cap \widehat{B_i} \cap \cdots \cap B_n) = \mathcal{V}$$

$$\square$$

Let $B \subseteq \mathcal{V}$ be any submodule whose spectrum is a disjoint union $A = A_1 \cup \cdots \cup A_n$ of closed subsets $A_i \subseteq M$. Then according to Lemma 1.4, Lemma 1.3 and Lemma 1.5 one has

$$(B + \mathfrak{n}_{A_1}\mathcal{V}) \cap \cdots \cap (B + \mathfrak{n}_{A_n}\mathcal{V}) = B + \mathfrak{n}_A \mathcal{V} = B$$

We observe that (by Lemma 1.6) $\mathrm{sp}(B + \mathfrak{n}_{A_i}\mathcal{V}) \subseteq \mathrm{sp}(B) \cap A_i = A_i$ while on the other hand $A = \mathrm{sp}(B) = \bigcup \mathrm{sp}(B + \mathfrak{n}_{A_i}\mathcal{V})$ so that $\mathrm{sp}(B + \mathfrak{n}_{A_i}\mathcal{V}) = A_i$ and the submodules $B + \mathfrak{n}_{A_i}\mathcal{V}$ are in general position. We will now prove the uniqueness of such a decomposition.

LEMMA 1.8. *Let $B_1, \ldots, B_n$ be submodules of $\mathcal{V}$ with disjoint spectra $A_1, \ldots, A_n$, let $B = B_1 \cap \cdots \cap B_n$. Then $B_i = B + \mathfrak{n}_{A_i}\mathcal{V}$.*

PROOF. We have

$$B^i := B + \mathfrak{n}_{A_i}\mathcal{V} \subseteq B_i + \mathfrak{n}_{A_i}\mathcal{V} = B_i$$

Suppose now that $B^j \subsetneqq B_j$ for some j. Because B^i are in general position we get

$$B^j + (B^1 \cap \cdots \cap \widehat{B^j} \cap \cdots \cap B^n) = \mathcal{V}$$

and therefore

$$B = B^1 \cap \cdots \cap B^n \subsetneqq B_j \cap (B^1 \cap \cdots \cap \widehat{B^j} \cap \cdots \cap B^n) \subseteq B_1 \cap \cdots \cap B_n = B$$

a contradiction. $\qquad\square$

We will now apply these ideas to submodules of finite codimension over $\mathbb{R}$. We fix an integer d and denote the collection of all the submodules of codimension d by $\mathrm{M}_d = \mathrm{M}_d(\mathcal{V})$. First observe that the spectrum of any such submodule is finite. This is obvious for ideals $I \subseteq \mathcal{R}$ as for any collection $y_1, \ldots, y_n \in \mathrm{sp}(I)$ of distinct points one can find functions $f_1, \ldots, f_n$ such that $f_i(y_i) = 1$ and $f_i(y_j) = 0$ if $i \neq j$. The f_i's are obviously linearly independent and hence their span is an n-dimensional linear subspace that clearly intersect I trivially. If $B \in \mathrm{M}_d(\mathcal{V})$ then

$$(B : \mathcal{V}) = \ker(\mathcal{R} \to \mathsf{End}(\mathcal{V}/B))$$

is an ideal of finite codimension and hence $\mathrm{sp}(B) = \mathrm{sp}(B : \mathcal{V})$ is finite.

Denoting the spectrum of B by $Y = \{y_1, \ldots, y_n\}$ we get the decomposition

$$B = (B + \mathfrak{n}_{y_1}\mathcal{V}) \cap \cdots \cap (B + \mathfrak{n}_{y_n}\mathcal{V})$$

where the submodules $B + \mathfrak{n}_{y_1}\mathcal{V}, \ldots, B + \mathfrak{n}_{y_n}\mathcal{V}$ are in general position and hence

$$\dim \mathcal{V}/B = \dim \mathcal{V}/(B + \mathfrak{n}_{y_1}\mathcal{V}) + \cdots + \dim \mathcal{V}/(B + \mathfrak{n}_{y_n}\mathcal{V})$$

We set $k_i = \dim \mathcal{V}/(B + \mathfrak{n}_{y_i}\mathcal{V})$ and we see that the spectrum of B has more structure if B has finite codimension: each point y_i in the spectrum has associated a weight k_i with it.

We define a space $\mathrm{S}_d(M)$ (where these more structured spectra will be defined) by

$$\mathrm{S}_d(M) := M^d/\Sigma_d$$

and give it the quotient topology. Very often we abbreviate $\mathrm{S}_d(M)$ to S_d. The space S_d consists of unordered collections of d not necessarily distinct points in M. If $y_1, \ldots, y_n$ are all the points in such a collection Y and if each y_i appears in it exactly k_i-times then we say that k_i is the weight of y_i and use an alternative notation

$$Y = \{(y_1, k_1), \ldots, (y_n, k_n)\}$$

We call $|Y| := \{y_1, \ldots, y_n\}$ the *support* of Y. There is a weight function $|Y| \to \mathbb{Z}^+$ associating to each point y_i its weight k_i and by the definition the total weight $\sum k_i$ is d. Therefore we also call Y a set of points with weights.

Hence with every submodule $B \subseteq \mathcal{V}$ of finite codimension d there is associated a canonical set of points with weights $Y \in \mathrm{S}_d$ whose support is the spectrum of B (and whose total weight is d). It is called the *weighted spectrum* of B.

On the other hand if $Y = \{(y_1, k_1), \ldots, (y_n, k_n)\} \in \mathrm{S}_d$ we define an ideal $\mathfrak{m}_Y \subseteq \mathcal{R}$ by

$$\mathfrak{m}_Y = (\mathfrak{m}_{y_1})^{k_1} \cdots (\mathfrak{m}_{y_n})^{k_n} = (\mathfrak{m}_{y_1})^{k_1} \cap \cdots \cap (\mathfrak{m}_{y_n})^{k_n}$$

(with the equality guaranteed by Lemma 1.4) and hence also get submodules $\mathfrak{m}_Y\mathcal{V} \subseteq \mathcal{V}$. They need not be of finite codimension unless $\mathcal{V}$ is finitely generated. We will see later that for a finitely generated $\mathcal{V}$ they do have a finite codimension.

LEMMA 1.9. *Let $B \subseteq \mathcal{V}$ be a submodule of finite codimension with weighted spectrum Y. Then $\mathfrak{m}_Y\mathcal{V} \subseteq B$.*

PROOF. Let us first prove the lemma in the case $Y = \{(y, k)\}$. We set $I = (B : \mathcal{V})$ and note that by our assumptions I is contained only in one maximal ideal, namely in $\mathfrak{m}_y$. We have a k-dimensional $\mathcal{R}/I$-module $\mathcal{V}/B$ and hence

$$(\mathfrak{m}_y)^{k+1}(\mathcal{V}/B) = (\mathfrak{m}_y)^k(\mathcal{V}/B)$$

for dimensional reasons. Now we apply Nakayama's lemma (see for example [**Lan**]) to conclude that $(\mathfrak{m}_y)^k(\mathcal{V}/B) = 0$, i.e. $(\mathfrak{m}_y)^k\mathcal{V} \subseteq B$.

In general when $|Y| = \{y_1, \ldots, y_n\}$ we use the decomposition

$$B = (B + \mathfrak{n}_{y_1}\mathcal{V}) \cap \cdots \cap (B + \mathfrak{n}_{y_n}\mathcal{V})$$

and Lemma 1.4 to reduce the proof to the case $n = 1$. $\qquad\square$

LEMMA 1.10. *For* $(\mathfrak{m}_y)^k$ *the following holds*

$$(\mathfrak{m}_y)^k = \{f \in \mathcal{R} \mid j_y^{k-1} f = 0\}$$

and it is a finitely generated ideal.

PROOF. The inclusion $\subseteq$ is obvious from the product formula for the derivative. Hence let $f \in \mathcal{R}$ be such that $j_y^{k-1} f = 0$. Let $\lambda : M \to \mathbb{R}$ be a function which is supported in a coordinate neighborhood around y and is identically 1 on a neighborhood of y. Then

$$f = \lambda \cdot f + (1 - \lambda) \cdot f$$

where the second summand is in $\mathfrak{n}_y \subseteq (\mathfrak{m}_y)^k$ so it remains to show that the first summand $g = \lambda \cdot f$ lies in the same. Note that $j_y^{k-1} g = 0$ as f and g agree near y and thus we can write in the coordinate chart

$$g(x) = \frac{1}{k!} \sum_{i_1,\ldots i_k=1}^{\dim M} a_{i_1 \ldots i_k}(x) \cdot x_{i_1} \cdots x_{i_k}$$

with $a_{i_1 \ldots i_k}$ smooth. If $\rho : M \to \mathbb{R}$ is any function that is identically 1 on a neighborhood of $\mathrm{supp}(\lambda)$ and is supported in the above coordinate chart we get smooth functions $\hat{x}_i = \rho \cdot x_i$ extended by 0 to M and similarly $\hat{a}_{i_1 \ldots i_k}$. Clearly on the whole whole of M we have

$$g(x) = \frac{1}{k!} \sum_{i_1,\ldots i_k=1}^{\dim M} \hat{a}_{i_1 \ldots i_k}(x) \cdot \hat{x}_{i_1} \cdots \hat{x}_{i_k} \in (\mathfrak{m}_y)^k$$

and moreover we see that $(\mathfrak{m}_y)^k$ is generated as an ideal by $1 - \lambda$ and the functions $\hat{x}_{i_1} \cdots \hat{x}_{i_k}$. $\square$

NOTE. Last lemma is not true in the C^r case. As an example, let $M = \mathbb{R}$ and $r = 1$, then any element of $\mathfrak{m}_0$ can be written in the form $f(x) = a(x)x$ with $a(x)$ continuous. Hence any element of $(\mathfrak{m}_0)^2$ can be written in the form $f(x) = a(x)x^2$ with $a(x)$ continuous again. In particular for any such function the limit

$$\lim_{x \to 0} \frac{f(x)}{x^2} = \lim_{x \to 0} a(x) = a(0)$$

exists. Certainly, the function $g(x) = x|x|$ is C^1 and

$$g(0) = g'(0) = 0$$

but

$$\lim_{x \to 0} \frac{g(x)}{x^2}$$

does not exist.

LEMMA 1.11. *For any set of points with weights* $Y = \{(y_1, k_1), \ldots, (y_n, k_n)\}$ *the following holds*

$$\mathfrak{m}_Y = \{f \in \mathcal{R} \mid j_{y_1}^{k_1-1} f = \cdots = j_{y_n}^{k_n-1} f = 0\}$$

In other words $\mathfrak{m}_Y$ *is the kernel of*

$$(j_{y_1}^{k_1-1}, \ldots, j_{y_n}^{k_n-1})^T : \mathcal{R} \longrightarrow J_{y_1}^{k_1-1}(M, \mathbb{R}) \times \cdots \times J_{y_n}^{k_n-1}(M, \mathbb{R})$$

Moreover it has a finite codimension in $\mathcal{R}$ *and it is finitely generated as an ideal.* $\square$

To proceed further we need to specialize to the case of finitely generated $\mathcal{V}$. If E is a finite dimensional real vector space, it induces a free $\mathcal{R}$-module $\mathcal{R} \otimes E \cong C^\infty(M, E)$ which we will abbreviate to $\mathcal{R}E$. We will call $\mathcal{V}$ *topologically finitely generated* if there is a surjective $\mathcal{R}$-module map $\mathcal{R}E \to \mathcal{V}$ that is a continuous quotient map, i.e. such that the topology on $\mathcal{V}$ is the quotient topology induced by this map. This notion will be important later.

COROLLARY 1.12. *If $\mathcal{V}$ is finitely generated then so is every $B \in \mathrm{M}_d$.*

PROOF. Let Y be the weighted spectrum of B. It is clear that when $\mathcal{V}$ is finitely generated then $\mathfrak{m}_Y \mathcal{V}$ has a finite codimension (an easy consequence of the case $\mathcal{V} = \mathcal{R}$ from Lemma 1.11). The statement then follows from the fact that $\mathfrak{m}_Y \mathcal{V}$ is finitely generated and its codimension in B is finite. $\square$

Let D be a linear subspace of an $\mathcal{R}$-module $\mathcal{V}$. We write $D \pitchfork \mathrm{M}_d(\mathcal{V})$ or just $D \pitchfork \mathrm{M}_d$ if D is transverse to every element of M_d, i.e. to every submodule of $\mathcal{V}$ of codimension d, and $D \pitchfork \mathrm{S}_d(\mathcal{V})$ or just $D \pitchfork \mathrm{S}_d$ if D is transverse to all submodules $\mathfrak{m}_Y \mathcal{V}$, whenever $Y \in \mathrm{S}_d$.

If $\mathcal{V}$ is finitely generated we construct a finite dimensional linear subspace $D \pitchfork \mathrm{S}_d$ (and hence also $D \pitchfork \mathrm{M}_d$). First we find this subspace locally in the case $\mathcal{V} = \mathcal{R}$.

LEMMA 1.13. *Let $M = \mathbb{R}^m$. There is a finite dimensional linear subspace $D \subseteq \mathcal{R}$ satisfying $D \pitchfork \mathrm{S}_d$.*

PROOF. Let D be the linear subspace of $\mathcal{R}$ of all polynomials $\mathbb{R}^m \to \mathbb{R}$ of degree at most $d - 1$ and let $Y \in \mathrm{S}_d$ be any set of points with weights. As the projection $\mathcal{R} \to \mathcal{R}/\mathfrak{m}_Y$ can be identified with the jet evaluation (see Lemma 1.11)

$$\mathcal{R} \longrightarrow J_{y_1}^{k_1 - 1}(\mathbb{R}^m, \mathbb{R}) \times \cdots \times J_{y_n}^{k_n - 1}(\mathbb{R}^m, \mathbb{R}) =: J_Y(\mathbb{R}^m, \mathbb{R})$$

we need to show that the composition

$$D \subseteq \mathcal{R}E \to J_Y(\mathbb{R}^m, \mathbb{R})$$

is surjective. We identify the jet spaces $J_{y_i}^{k_i - 1}(\mathbb{R}^m, \mathbb{R})$ with the truncated polynomial algebra of polynomials $\mathbb{R}^m \to \mathbb{R}$ of degree at most $k_i - 1$ in such a way that the polynomial p corresponds to the jet $j_{y_i}^{k_i - 1} p(x - y_i)$. This translation ensures that the identification is an isomorphism of algebras. To prove surjectivity let

$$(0, \ldots, 0, p_i, 0, \ldots, 0) \in J_Y(\mathbb{R}^m, \mathbb{R})$$

We choose a linear form $\alpha : \mathbb{R}^m \to \mathbb{R}$ which is injective on the support $|Y|$ and consider the following polynomial

$$q(x) = \prod_{j \neq i} (\alpha(x) - \alpha(y_j))^{k_j}$$

By construction $q(y_i) \neq 0$ and thus $q(x + y_i)$ is invertible in the truncated polynomial algebra. Hence we can find a polynomial $r(x)$ of degree at most $k_i - 1$ such that $q(x + y_i)r(x) = p_i(x)$. This product corresponds to the jet $j_{y_i}^{k_i - 1}(q(x)r(x - y_i))$ and so $q(x)r(x - y_i) \in D$ is a preimage of $(0, \ldots, 0, p_i, 0, \ldots, 0)$. $\square$

THEOREM 1.14. *Let M be any compact manifold. Then there is a finite dimensional linear subspace $D \subseteq \mathcal{R}$ for which $D \pitchfork \mathrm{S}_d$.*

PROOF. Let us choose an embedding $\iota : M \hookrightarrow V$ of M into a Euclidean space V and note that the canonical map

$$\iota^* : C^\infty(V, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$$

is surjective (i.e. any smooth map $M \to \mathbb{R}$ is extensible to V). By Lemma 1.13 there exists a finite dimensional linear subspace $D \subseteq C^\infty(V, \mathbb{R})$ such that $D \pitchfork \mathrm{S}_d(V)$. Then we claim that $\iota^* D = \{f\iota \mid f \in D\}$ satisfies $\iota^* D \pitchfork \mathrm{S}_d(M)$. This is verified by the commutative diagram

$$
\begin{array}{ccccc}
D & \lhook\joinrel\longrightarrow & C^\infty(V, \mathbb{R}) & \longrightarrow\mkern-14mu\rightarrow & J_{\iota(Y)}(V, \mathbb{R}) \\
\downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\
\iota^* D & \lhook\joinrel\longrightarrow & C^\infty(M, \mathbb{R}) & \longrightarrow\mkern-14mu\rightarrow & J_Y(M, \mathbb{R})
\end{array}
$$

The composition across the top row is surjective and therefore so must be the composition across the bottom one. $\qquad\square$

THEOREM 1.15. *Let M be a compact manifold and let $\mathcal{V}$ be a finitely generated $\mathcal{R}$-module. Then there is a finite dimensional linear subspace $D \subseteq \mathcal{V}$ for which $D \pitchfork \mathrm{S}_d$.*

PROOF. First we give a proof in the special case $\mathcal{V} = \mathcal{R}E$ of a free $\mathcal{R}$-module of finite rank. By Theorem 1.14 there exists a finite dimensional linear subspace $D' \subseteq \mathcal{R}$ such that $D' \pitchfork \mathrm{S}_d$. Clearly

$$\hat{D} := D' \otimes E \subseteq \mathcal{R} \otimes E = \mathcal{R}E$$

does the job as $\mathfrak{m}_Y \mathcal{R}E = \mathfrak{m}_Y \otimes E$. In the general case let $\varphi : \mathcal{R}E \to \mathcal{V}$ be any surjective map of $\mathcal{R}$-modules. As we just saw there is a $\hat{D} \subseteq \mathcal{R}E$ such that $\hat{D} \pitchfork \mathrm{S}_d(\mathcal{R}E)$. We claim now that $\varphi(\hat{D}) \pitchfork \mathrm{S}_d(\mathcal{V})$. Hence let $Y \in \mathrm{S}_d$. Then

$$B = \varphi^{-1}(\mathfrak{m}_Y \mathcal{V}) = \mathfrak{m}_Y \mathcal{R}E + \ker \varphi$$

and so $\hat{D} + B = \mathcal{R}E$. Therefore

$$\varphi(\hat{D}) + \mathfrak{m}_Y \mathcal{V} = \mathcal{V}$$

and indeed $D = \varphi(\hat{D}) \pitchfork \mathrm{S}_d(\mathcal{V})$. $\qquad\square$

Let us fix a finite dimensional linear subspace $D \subseteq \mathcal{V}$. Then there is the following adjoint pair of functors between posets

$$\{\text{linear subspaces of } D\} \underset{U_D}{\overset{F_D}{\rightleftarrows}} \{\mathcal{R}\text{-submodules of } \mathcal{V}\}$$

where F_D maps $L \subseteq D$ to the $\mathcal{R}$-submodule $\mathcal{R}L$ generated by L while U_D maps $B \subseteq \mathcal{V}$ to $B \cap D$.

PROPOSITION 1.16. *If $D \pitchfork \mathrm{M}_{d+1}$ then $F_D U_D|_{\mathrm{M}_d} = \mathrm{id}$. In particular $U_D|_{\mathrm{M}_d}$ is injective and if B' and B'' are submodules of codimension at most d then*

$$B' \subseteq B'' \quad \Leftrightarrow \quad D \cap B' \subseteq D \cap B''$$

Proof. Let $B \in M_d$. For brevity we denote

$$\hat{B} = \mathcal{R}(D \cap B) = F_D U_D(B)$$

and let us assume that $\hat{B} \subsetneq B$. We choose a maximal submodule $\tilde{B}$ with

$$\hat{B} \subseteq \tilde{B} \subsetneq B$$

which is possible by the finite generation of B. By maximality $B/\tilde{B} \cong \mathcal{R}/\mathfrak{m}_y$ for some maximal ideal $\mathfrak{m}_y$ and in particular $\tilde{B} \in M_{d+1}$, hence $D \pitchfork \tilde{B}$. By transversality then $D \cap \hat{B} \subseteq D \cap \tilde{B} \subsetneq D \cap B$. This is a contradiction with the triangle identity $U_D F_D U_D = U_D$ for the adjoint pair (F_D, U_D).

The second claim follows for in that case $B' = F_D U_D(B')$, $B'' = F_D U_D(B'')$ and both F_D and U_D preserve inclusions. □

Remark. Mere $D \pitchfork M_d$ would not be enough for injectivity of U_D as the example of polynomials $\mathbb{R}^m \to \mathbb{R}$ of degree at most $d-1$ shows.

Proposition 1.17. *If $D \subseteq D'$ are finite dimensional linear subspaces of $\mathcal{V}$ such that $D \pitchfork M_d$ then the map*

$$U_{D'} M_d \longrightarrow U_D M_d$$

sending a subspace $L \subseteq D'$ to its intersection with D is continuous where both spaces are given the subspace topology from the respective Grassmannian manifolds of subspaces of codimension d. Also the map of the canonical d-dimensional bundles (with fibres D'/L over L and corresponding fibre $D/(D \cap L)$ over the image $D \cap L$) is continuous.

Proof. Let $V \subseteq G_d(D')$ be the subset of subspaces of D' of codimension d transverse to D. Then the map $V \to G_d(D)$ sending L to $D \cap L$ is continuous (even smooth) and the map from the statement is just its restriction to the subset $U_{D'} M_d$. The same works for the canonical bundles. □

1.2. The topology on M_d

It turns out that one of the most important tools in the proofs in this section is an existence of a continuous interpolation map. It is a map $A : \mathcal{R} \times S_d \to D$ (for some $D \pitchfork S_d$) with the property $A(f, Y) = f$ modulo $\mathfrak{m}_Y$. By definition for each $f \in \mathcal{R}$ and $Y \in S_d$ such an $A(f, Y)$ exists but there is no obvious choice and in particular it is not obvious that there is a continuous way of choosing it. For example if D is the space of polynomials of degree at most $d-1$, we constructed an interpolation in the proof of Lemma 1.13. However as the following example shows it fails to be continuous.

Example 1.18. Let $M = \mathbb{R}^2$, $Y_n = [(0,0), (-1/n, 1/n), (1/n, 1/n)]$. Let us consider the function $f(x, y) = y$ and let us construct the interpolations in the space D of truncated polynomials of degree at most 2 by the method from the proof of Lemma 1.13 with respect to $\alpha(x, y) = x$ which clearly is injective on each Y_n. Easily

$$p_n(x, y) = 0 \cdot (x + 1/n)(x - 1/n) + n/2 \cdot x(x - 1/n) + n/2 \cdot x(x + 1/n) = nx^2$$

and therefore p_n does not converge in D. In fact there is no choice of α which would produce a convergent sequence. $\qquad\square$

In the last example it is very easy to produce a convergent interpolation sequence (after all $f \in D$, so we can as well take $p_n = f$). The following construction is one way how to construct canonical interpolations in the local case $M = \mathbb{R}^m$.

Let $f : \mathbb{R}^m \to E$ be a smooth function. If $(x_0, \ldots, x_r) \in (\mathbb{R}^m)^{r+1}$ we denote by $[x_0, \ldots, x_r] : \Delta^r \to \mathbb{R}^m$ the unique affine map sending the vertices of the standard r-simplex Δ^r to $x_0, \ldots, x_r$. It is obvious that this gives a bijective correspondence between $(\mathbb{R}^m)^{r+1}$ and affine maps $\Delta^r \to \mathbb{R}^m$. We will denote a general affine map $\Delta^r \to \mathbb{R}^m$ by σ, if we do not want to emphasize the values at vertices. By embedding it linearly into $\mathbb{R}^r$ we give Δ^r the Lebesgue measure in which the volume is 1. Then we are able to define unambiguously

$$\mathcal{I}(f, \sigma) \in \mathrm{Hom}(S^r \mathbb{R}^m, E)$$

a symmetric r-form on $\mathbb{R}^m$ with values in E to be

$$\mathcal{I}(f, \sigma) = \int_{\Delta^r} f^{(r)} \sigma$$

where $f^{(r)} : \mathbb{R}^m \to \mathrm{Hom}(S^r \mathbb{R}^m, E)$ denotes the r-fold derivative of f. In an obvious way by omitting x_i we get $\partial_i \sigma : \Delta^{r-1} \to \mathbb{R}^m$ and thus forms

$$\mathcal{I}(f, \partial_i \sigma) \in \mathrm{Hom}(S^{r-1} \mathbb{R}^m, E)$$

LEMMA 1.19. *For any smooth function $g : \mathbb{R}^m \to E$ and for any $\sigma = [x_0, \ldots, x_r]$, the following holds for $0 \le i, j \le r$*

$$\int_{\Delta^r} g'_{x_j - x_i} \sigma = r \left(\int_{\Delta^{r-1}} g(\partial_i \sigma) - \int_{\Delta^{r-1}} g(\partial_j \sigma) \right)$$

where $g'_{x_j - x_i}$ denotes the derivative of g in the direction $x_j - x_i$. In particular by taking $g = f^{(r-1)}$ we have

$$\mathcal{I}(f, \sigma)(v_1, \ldots, v_{r-1}, x_j - x_i) = r(\mathcal{I}(f, \partial_i \sigma) - \mathcal{I}(f, \partial_j \sigma))(v_1, \ldots, v_{r-1})$$

PROOF. Define a map $\delta : \Delta^{r-2} \to \Delta^r$ to be

$$[e_0, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_r]$$

where e_n are the vertices of Δ^r. We think of Δ^{r-1} as a convex span of Δ^{r-2} (with vertices $e_1, \ldots, e_{r-1}$) and an additional point e_0. Then we can define a homotopy

$$h : \Delta^{r-1} \times I \to \Delta^r$$

by a formula (with x running over Δ^{r-2})

$$h(t_0 e_0 + (1 - t_0)x, t) = t_0((1 - t)e_i + t e_j) + (1 - t_0)\delta(x)$$

One sees easily that $h(-, 0)$ is the inclusion of the j-th face of Δ^r and $h(-, 1)$ the inclusion of the i-th one. To compute the determinant of h' we choose a basis $(e_1 - e_0, \ldots, e_{r-1} - e_0, e)$ of $T(\Delta^{r-1} \times I)$ where e is the unit tangent vector of I. Then

$$h'(t_0 e_0 + (1 - t_0)x, t)(e_n - e_0) = \delta(e_n) - e_i - t(e_j - e_i)$$

and

$$h'(t_0 e_0 + (1 - t_0)x, t)(e) = t_0(e_j - e_i)$$

Hence we easily get a formula

$$|\det h'(t_0 e_0 + (1 - t_0)x, t)| = ct_0$$

for some constant c and it is not difficult to see that $c = r$. Then

$$\int_{\Delta^r} g'_{x_j - x_i} \sigma = \int_{\Delta^{r-1} \times I} rt_0 g'_{x_j - x_i} \sigma h = r \int_{\Delta^{r-1}} \int_0^1 t_0 g'_{x_j - x_i} \sigma h(-, t) \, \mathrm{d}t$$

Now note that

$$\frac{\partial}{\partial t}(g\sigma h) = t_0 g'_{x_j - x_i} \sigma h$$

and so

$$\int_{\Delta^{r-1}} \int_0^1 t_0 g'_{x_j - x_i} \sigma h(-, t) \mathrm{d}t \;\; = \;\; \int_{\Delta^{r-1}} g\sigma h(-, 1) - \int_{\Delta^{r-1}} g\sigma h(-, 0)$$

$$= \;\; \int_{\Delta^{r-1}} g(\partial_i \sigma) - \int_{\Delta^{r-1}} g(\partial_j \sigma)$$

$\square$

COROLLARY 1.20. *The following formula holds*

$$f(x_r) = \mathcal{I}(f, [x_0]) + \cdots + \frac{1}{i!} \cdot \mathcal{I}(f, [x_0, \ldots, x_i])(x_r - x_0, \ldots, x_r - x_{i-1}) + \cdots$$
$$+ \frac{1}{r!} \cdot \mathcal{I}(f, [x_0, \ldots, x_r])(x_r - x_0, \ldots, x_r - x_{r-1}) \tag{1.1}$$

PROOF. We use induction with respect to r. According to the previous lemma

$$\frac{1}{r!} \cdot \mathcal{I}(f, [x_0, \ldots, x_r])(x_r - x_0, \ldots, x_r - x_{r-1}) \tag{1.2}$$

is equal to

$$\frac{1}{(r-1)!} \cdot (\mathcal{I}(f, [x_0, \ldots, \widehat{x_{r-1}}, x_r]) - \mathcal{I}(f, [x_0, \ldots, x_{r-1}]))(x_r - x_0, \ldots, x_r - x_{r-2})$$

Adding the remaining terms of (1.1) to (1.2) and using the inductive hypothesis on $[x_0, \ldots, \widehat{x_{r-1}}, x_r]$ we prove the inductive step. $\square$

COROLLARY 1.21. *The following conditions are equivalent*

(i) $\mathcal{I}(f, \tau) = 0$ *for all the faces τ of σ.*
(ii) $\mathcal{I}(f, \partial_1 \cdots \partial_r \sigma) = \cdots = \mathcal{I}(f, \partial_i \cdots \partial_r \sigma) = \cdots = \mathcal{I}(f, \partial_r \sigma) = \mathcal{I}(f, \sigma) = 0.$

PROOF. We assume (ii). By induction we can also assume that for all the faces τ of $\partial_r \sigma$, we have $\mathcal{I}(f, \tau) = 0$. By the previous lemma $\mathcal{I}(f, \partial_i \sigma) = 0$ for all i. As there is a common face of $\partial_i \sigma$ and $\partial_r \sigma$ we see that up to a renumbering of vertices the condition (ii) is satisfied for $\partial_i \sigma$ and by induction again we get (i) for all the faces of $\partial_i \sigma$. $\square$

COROLLARY 1.22. *Let $\sigma = [x_0, \ldots, x_r]$. If*

$$\mathcal{I}(f, \partial_1 \cdots \partial_r \sigma) = \cdots = \mathcal{I}(f, \partial_i \cdots \partial_r \sigma) = \cdots = \mathcal{I}(f, \partial_r \sigma) = \mathcal{I}(f, \sigma) = 0$$

and if y appears k-times in $x_0, \ldots, x_r$ then $j_y^{k-1} f = 0$. $\qquad\square$

REMARK. The converse is not true in general (with the exception $x_0 = \cdots = x_r$) for dimensional reasons unless $m = 1$: if $\{(y_1, k_1), \ldots, (y_n, k_n)\}$ denotes the class of $(x_0, \ldots, x_r)$ in $\mathrm{S}_{r+1}(\mathbb{R}) = \mathbb{R}^{r+1}/\Sigma_{r+1}$ then $j_{y_i}^{k_i-1} f = 0$ for all $i = 1, \ldots, n$ implies $\mathcal{I}(f, [x_0, \ldots, x_j]) = 0$ for all $j = 0, \ldots, r$.

Now we will explain how this leads to an interpolation map. First we restrict ourselves to interpolation at points close to a single point, later generalizing to a number of points. This is only to lighten the notation a bit. Going back from the local situation to the case of a compact manifold M we consider $\mathcal{V} = \mathcal{R}E = C^\infty(M, E)$, a free $\mathcal{R}$-module of a finite rank. We fix $y \in M$ and $k \geq 1$ and identify a neighborhood of y with $\mathbb{R}^m$. We also fix a complementary linear subspace F to the submodule $(\mathfrak{m}_y)^k \mathcal{V}$ and define the following map

$$G : \mathcal{V} \times (\mathbb{R}^m)^k \longrightarrow \mathrm{Hom}(S^0\mathbb{R}^m, E) \times \cdots \times \mathrm{Hom}(S^{k-1}\mathbb{R}^m, E) \times (\mathbb{R}^m)^k$$
$$\cong J_*^{k-1}(\mathbb{R}^m, E) \times (\mathbb{R}^m)^k$$

(with $J_*^{k-1}(\mathbb{R}^m, E)$ being any $J_x^{k-1}(\mathbb{R}^m, E)$ – they are all identified via translations) by the formula

$$G(f, (y_1, \ldots, y_k)) = (\mathcal{I}(f, [y_1]), \ldots, \mathcal{I}(f, [y_1, \ldots, y_k]), (y_1, \ldots, y_k))$$

We denote by G_F its restriction

$$G_F : F \times (\mathbb{R}^m)^k \to J_*^{k-1}(\mathbb{R}^m, E) \times (\mathbb{R}^m)^k$$

Note that for each $f \in \mathcal{V}$ the map $G(f, -)$ is continuous (in fact smooth) and therefore so is G_F. Our transversality condition on F implies that on the fibres over $(y, \ldots, y)$

$$(G_F)_{(y, \ldots, y)} : F \to J_*^{k-1}(\mathbb{R}^m, E)$$

is a linear isomorphism. Hence we find a neighborhood of $(y, \ldots, y)$ in M^k of the form U^k with U compact convex, such that the restriction

$$G_F : F \times U^k \to J_*^{k-1}(\mathbb{R}^m, E) \times U^k$$

is an isomorphism of vector bundles over U^k. Hence we can define a map

$$\hat{A} : \mathcal{V} \times U^k \xrightarrow{G} J_*^{k-1}(\mathbb{R}^m, E) \times U^k \xrightarrow{G_F^{-1}} F \times U^k \to F$$

Now note that according to Corollary 1.21 the value of $\hat{A}$ does not depend on the ordering of the points and hence we get

$$A : \mathcal{V} \times U^k/\Sigma_k \to F$$

with the property that $A(f, Y)$ is an interpolation of f at Y, i.e. such that $f = A(f, Y)$ modulo $\mathfrak{m}_Y \mathcal{V}$.

THEOREM 1.23. *The interpolation map A is continuous and $A(f, \{(y, k)\}) = 0$ whenever $f \in (\mathfrak{m}_y)^k \mathcal{V}$.*

PROOF. By the construction of A it is enough to show the continuity of each component

$$G_i : \mathcal{V} \times U^k \to \mathrm{Hom}(S^i \mathbb{R}^m, E)$$

of G, $i = 0, \ldots, k-1$. Hence let us fix $(f, (y_1, \ldots, y_k)) \in \mathcal{V} \times U^k$ and denote $Y = (y_1, \ldots, y_k)$ for a short. We choose a norm on $\mathrm{Hom}(S^i \mathbb{R}^m, E)$ and a neighborhood

$$\{\alpha \in \mathrm{Hom}(S^i \mathbb{R}^m, E) \mid \|\alpha - G_i(f, Y)\| < \varepsilon\}$$

Because of the continuity of $G_i(f, -)$ there is a neighborhood V of Y on which

$$\|G_i(f, -) - G_i(f, Y)\| < \varepsilon/2$$

Hence if g is such that $\|g^{(i)} - f^{(i)}\| < \varepsilon/2$ on U then also

$$\|G_i(g, -) - G_i(f, -)\| < \varepsilon/2$$

on U and finally on V we have

$$\|G_i(g, -) - G_i(f, Y)\| \leq \|G_i(g, -) - G_i(f, -)\| + \|G_i(f, -) - G_i(f, Y)\| < \varepsilon$$

Because the condition on $g^{(i)}$ defines a neighborhood of f in $\mathcal{V}$, this finishes the proof. □

EXAMPLE 1.24. In the one dimensional local case $M = \mathbb{R}$ the space P_{k-1} of polynomials $\mathbb{R} \to \mathbb{R}$ of degree at most $k-1$ is clearly a complementary subspace of $(\mathfrak{m}_y)^k$. For any $(y_1, \ldots, y_k) \in \mathbb{R}^k$ consider the following basis of P_{k-1}:

$$\{1, (x - y_1), \ldots, 1/i! \cdot (x - y_1) \cdots (x - y_i), \ldots, 1/k! \cdot (x - y_1) \cdots (x - y_k)\}$$

Then one has

$$\mathcal{I}(1/i! \cdot (x - y_1) \cdots (x - y_i), [y_1, \ldots, y_j]) = \delta_{ij}$$

This is clear for $j \geq i$ by a direct computation and for $i > j$ this follows from the remark after Corollary 1.22. Consequently one obtains a formula

$$A(f, \Sigma_k(y_1, \ldots, y_k)) = \mathcal{I}(f, [y_1]) + \cdots + 1/(k-1)! \cdot \mathcal{I}(f, [y_1, \ldots, y_k])(x - y_1) \ldots (x - y_{k-1})$$

This is the so-called Lagrange's interpolation formula. It can be found for example in Section 4.2 of [**Sch**].

Now if we have an arbitrary $Y = \{(y_1, k_1,), \ldots, (y_n, k_n)\} \in S_d$ we identify a neighbourhood of each y_i with $\mathbb{R}^m$. Writing

$$\mathbb{R}^{md} \cong (\mathbb{R}^m)^{k_1} \times \cdots \times (\mathbb{R}^m)^{k_n}$$

we replace G by the corresponding map

$$G : \mathcal{V} \times \mathbb{R}^{md} \to J_*^{k_1-1}(\mathbb{R}^m, E) \times \cdots \times J_*^{k_n-1}(\mathbb{R}^m, E) \times \mathbb{R}^{md}$$

Again if F is a complementary linear subspace to $\mathfrak{m}_Y \mathcal{V}$ then on the fibres

$$(G_F)_{((y_1, \ldots, y_1), \ldots, (y_n, \ldots, y_n))} : F \to J_*^{k_1-1}(\mathbb{R}^m, E) \times \cdots \times J_*^{k_n-1}(\mathbb{R}^m, E)$$

is an isomorphism and we get a neighborhood of the form $U_1^{k_1} \times \cdots \times U_n^{k_n}$ over which G_F is an isomorphism. Denoting the induced neighborhood of Y in S_d by W one gets an interpolation map

$$A : \mathcal{V} \times W \to F$$

THEOREM 1.25. *The interpolation map A is continuous and $A(f, Y) = 0$ whenever $f \in \mathfrak{m}_Y \mathcal{V}$.* □

Next corollary is crucial for the proof of compactness of M_d.

COROLLARY 1.26. *Let M be a compact manifold, let $\mathcal{V} = \mathcal{R}E$, let $D \pitchfork S_d$. Let $Y_p \in S_d$ $(p = 1, 2, \ldots)$ be a sequence converging to Y. Then every $v \in D \cap \mathfrak{m}_Y \mathcal{V}$ is a limit of some sequence $v_p \in D \cap \mathfrak{m}_{Y_p} \mathcal{V}$.*

PROOF. One chooses a complementary linear subspace $F \subseteq D$ to $\mathfrak{m}_Y \mathcal{V}$ and gets a sequence $v_p = v - A(v, Y_p) \to v$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

THEOREM 1.27. *Let M be a compact manifold, let $\mathcal{V} = \mathcal{R}E$, let $D \pitchfork S_d$. Then there is a continuous fibrewise linear interpolation map*

$$A : \mathcal{V} \times S_d \to D$$

i.e. a map such that $A(f, Y) = f$ modulo $\mathfrak{m}_Y \mathcal{V}$.

PROOF. One glues the local interpolation maps using a partition of unity on S_d. Here one uses the fact that the interpolation maps form an affine space. $\qquad\square$

With Corollary 1.26 at hand we can prove the main theorem in the same way Glaeser did in [**Gla**] for a parallelepiped in $\mathbb{R}^m$.

THEOREM 1.28. *Let M be a compact manifold, let $\mathcal{V} = \mathcal{R}E$ be a free $\mathcal{R}$-module of finite rank and let $D \pitchfork S_{d+1}$. Then $U_D M_d \subseteq G_d(D)$ is a closed subset hence compact. Also the map $\pi : U_D M_d \to S_d$ sending $D \cap B$ to the weighted spectrum of B is continuous.*

PROOF. Let B_p be a sequence of submodules of codimension d and let $L_p = D \cap B_p$. Let us assume that L_p converges to L in $G_d(D)$. We will construct a submodule B such that $L = D \cap B$. By taking a subsequence we can assume that the sequence of weighted spectra corresponding to B_p converges to Y. We set $B = \mathfrak{m}_Y \mathcal{V} + L$. According to Corollary 1.26, $D \cap \mathfrak{m}_Y \mathcal{V} \subseteq L$ and this easily implies that $D \cap B = L$ and that the codimension of B is d. It remains to show that B is indeed a submodule.

We choose a finite dimensional linear subspace $P \subseteq \mathcal{R}$ complementary to $\mathfrak{m}_Y$. We can make sure that $1 \in P$. Then $\mathfrak{m}_Y + P = \mathcal{R}$ and to prove that B is a submodule one needs to prove the inclusion labeled by ? in

$$\mathcal{R}B = (\mathfrak{m}_Y + P)(\mathfrak{m}_Y \mathcal{V} + L) \subseteq \mathfrak{m}_Y \mathcal{V} + PL \overset{?}{\subseteq} \mathfrak{m}_Y \mathcal{V} + L = B$$

In order to do so one constructs an analogous sequence $PD \cap B_p$ in PD and by taking a further subsequence one can assume that this sequence converges to L'. As $1 \in P$ clearly $L \subseteq L'$ and so

$$\mathfrak{m}_Y \mathcal{V} + L \subseteq \mathfrak{m}_Y \mathcal{V} + L'$$

But the codimensions are the same (equal to d) and therefore these two subspaces must be equal. As $P(D \cap B_p) \subseteq PD \cap B_p$ taking the limit we get $PL \subseteq L'$ and so

$$\mathfrak{m}_Y \mathcal{V} + PL \subseteq \mathfrak{m}_Y \mathcal{V} + L' = \mathfrak{m}_Y \mathcal{V} + L$$

finishing the proof that B is a submodule.

To prove continuity of the map $U_D M_d \to S_d$ it is enough (by compactness of S_d) to show that the weighted spectrum of the submodule B that we just constructed is

indeed Y. The idea is to use the primary decomposition and compare the dimensions. We choose some disjoint closed neighborhoods C_i of y_i and decompose

$$B_p = (B_p + \mathfrak{n}_{C_1}\mathcal{V}) \cap \cdots \cap (B_p + \mathfrak{n}_{C_n}\mathcal{V})$$

By the convergence $\mathrm{sp}(B_p) \to Y$ we easily see that the codimension of each $B_p^i :=$ $B_p + \mathfrak{n}_{C_i}\mathcal{V}$ is k_i, at least for all big enough p. Assuming that each sequence $D \cap B_p^i$ converges in $G_{k_i}(D)$ to $D \cap B^i$ we certainly have

$$D \cap B = \lim(D \cap B_p^1) \cap \cdots \cap (D \cap B_p^n) \subseteq$$
$$\subseteq (D \cap B^1) \cap \cdots \cap (D \cap B^n) = D \cap (B^1 \cap \cdots \cap B^n)$$

 and by the transversality assumptions and Proposition 1.16 we conclude that $B \subseteq$ $B^1 \cap \cdots \cap B^n$. By the proof above the spectrum of each B^i consists just of y_i hence the B^i are in general position and therefore the codimension of $B^1 \cap \cdots \cap B^n$ is also d. Thus $B = B^1 \cap \cdots \cap B^n$ is the primary decomposition of B proving the claim about its weighted spectrum. $\qquad \square$

Hence by Proposition 1.17 we know that the topology on M_d does not depend on the choice of D as long as $D \pitchfork M_d$ and U_D is injective.

REMARK. Let us consider the case $\mathcal{V} = \mathcal{R}$ and for simplicity denote $M_d(M) = M_d(\mathcal{R})$. For an open subset $U \subseteq M$ we define $M_d(U) \subseteq M_d(M)$ by the pullback square

$$\begin{array}{ccc} M_d(U) & \hookrightarrow & M_d(M) \\ \downarrow & & \downarrow \pi \\ S_d(U) & \hookrightarrow & S_d(M) \end{array}$$

In other words $M_d(U)$ is the space of ideals in $\mathcal{R}$ with spectrum in U. Theorem 1.28 implies that the topology of $M_d(U)$ is independent of M and $M_d(M)$ is a union of $M_d(U)$ as U varies over coordinate charts. In other words $M_d(M)$ is local. On the other hand the glueing maps are not polynomial and so [**Gla**] could not be used.

Let $E_d(D)$ denote the canonical d-dimensional vector bundle over $G_d(D)$, whose fibre over $F \subseteq D$ is D/F. Then again by the same proposition the restriction $E_d(D)|_{U_D M_d}$ is independent of D and will be denoted by E_d.

COROLLARY 1.29. *Let M be a compact manifold and let $\mathcal{V} = \mathcal{R}E$ be a free $\mathcal{R}$-module of finite rank. Then the natural map*

$$\mathcal{V} \times M_d \to E_d$$

defined on the fibre over B by

$$\mathcal{V} \to \mathcal{V}/B \cong D/(D \cap B)$$

is a continuous quotient map (in the topological sense) of vector bundles over M_d.

PROOF. The map in question can be defined alternatively as

$$\mathcal{V} \times M_d \xrightarrow{\mathrm{id}_{\mathcal{V}} \times (\pi, \mathrm{id}_{M_d})} \mathcal{V} \times S_d \times M_d \xrightarrow{A \times \mathrm{id}_{M_d}} D \times M_d \to E_d$$

using the interpolation map A constructed above and hence is by Theorem 1.27 continuous. Using an inner product on D one can easily find a section proving that it is a quotient map. $\square$

Corollary 1.30. *Let M be a compact manifold, let $\mathcal{V} = \mathcal{R}E$ be a free $\mathcal{R}$-module of finite rank and let $K \subseteq \mathcal{R}E$ be a submodule. Then the subset*

$$\{B \in M_d \mid K \subseteq B\} \subseteq M_d$$

is closed hence compact.

Proof. For each $v \in K$ the restriction of the canonical map

$$M_d \cong \{v\} \times M_d \subseteq \mathcal{V} \times M_d \to E_d$$

is continuous by the last corollary and the set $\{B \in M_d \mid v \in B\}$ is precisely the preimage of the zero section and therefore closed. So is then their intersection over all $v \in K$, the set from the statement. $\square$

Theorem 1.31. *Let M be a compact manifold, let $\mathcal{V}$ be any finitely generated $\mathcal{R}$-module and let $D \pitchfork M_d$. Then $U_D M_d \subseteq G_d(D)$ is a closed subset hence compact. If $\mathcal{V}$ is topologically finitely generated then the natural map*

$$\mathcal{V} \times M_d \to E_d$$

is a continuous quotient map.

Proof. Let $\varphi : \mathcal{R}E \to \mathcal{V}$ be any surjective map of $\mathcal{R}$-modules with E a finite dimensional real vector space, let $K = \ker \varphi$. As we can replace D by any bigger subspace to prove compactness we can assume that $D = \varphi(\hat{D})$ for some $\hat{D} \pitchfork M_{d+1}(\mathcal{R}E)$. The submodules of $\mathcal{V} \cong \mathcal{R}E/K$ are identified with those submodules of $\mathcal{R}E$ that contain K. Now we reconstruct this relation inside $\hat{D}$ and D: setting $\hat{K} := \hat{D} \cap K$ we see that $\varphi|_{\hat{D}} : \hat{D} \to D$ can be identified with the projection $\hat{D} \to \hat{D}/\hat{K}$. Consequently the map $L \mapsto \varphi(L)$ is continuous when restricted to subspaces containing $\hat{K}$ and so the dashed arrow in the diagram

$$
\begin{array}{ccccc}
U_{\hat{D}}\{B \mid K \subseteq B\} & \hookrightarrow & \{L \mid \hat{K} \subseteq L\} & \hookrightarrow & G_d\hat{D} \\
\downarrow & & \downarrow & & \\
U_D M_d(\mathcal{V}) & \hookrightarrow & G_d D & &
\end{array}
$$

is then a continuous bijection. As $U_{\hat{D}}\{B \mid K \subseteq B\}$ is compact by Corollary 1.30 this dashed arrow is a homeomorphism identifying $U_D M_d(\mathcal{V})$ with a subspace of $U_{\hat{D}} M_d(\mathcal{R}E)$. In the same way $E_d(\mathcal{V})$ can be thought of as the restriction of $E_d(\mathcal{R}E)$ to $M_d(\mathcal{V})$ and in the diagram

$$
\begin{array}{ccccc}
\mathcal{V} \times M_d(\mathcal{V}) & \twoheadleftarrow & \mathcal{R}E \times M_d(\mathcal{V}) & \hookrightarrow & \mathcal{R}E \times M_d(\mathcal{R}E) \\
& \searrow & \downarrow & & \downarrow \\
& & E_d(\mathcal{V}) & \hookrightarrow & E_d(\mathcal{R}E)
\end{array}
$$

the dashed arrow exists and is continuous by the properties of quotients. $\square$

We consider the space $G^r(\mathcal{V})$ of r-dimensional linear subspaces of $\mathcal{V}$ and give it the quotient topology from $V^r(\mathcal{V})/GL_r(\mathbb{R})$ where

$$V^r(\mathcal{V}) \subseteq \mathcal{V} \times \cdots \times \mathcal{V}$$

is the subset of linearly independent r-tuples of vectors in $\mathcal{V}$ and the action of $GL_r(\mathbb{R})$ is the usual one.

COROLLARY 1.32. *The set of all r-dimensional linear subspaces D of $\mathcal{V}$ for which $D \pitchfork M_d$ is open in $G^r(\mathcal{V})$.*

PROOF. As the canonical projection $V^r(\mathcal{V}) \to G^r(\mathcal{V})$ is (defined to be) a quotient map we only need to show that the preimage $S \subseteq V^r(\mathcal{V})$ of the set from the statement is open. We consider the map

$$\alpha : \mathcal{V} \times \cdots \times \mathcal{V} \times M_d \longrightarrow E_d \times_{M_d} \cdots \times_{M_d} E_d \cong \mathrm{Hom}(\mathbb{R}^r, E_d)$$

sending $((v_1, \ldots, v_r), B)$ to the homomorphism $\mathbb{R}^r \to \mathcal{V}/B$ which is determined by sending the basis vectors to the classes of the v_i's in $\mathcal{V}/B$. According to the previous theorem α is continuous. The subset $\mathrm{SurHom}(\mathbb{R}^r, E_d)$ of surjective homomorphisms is open in the target and hence so is

$$T := (V^r(\mathcal{V}) \times M_d) \cap \alpha^{-1}(\mathrm{SurHom}(\mathbb{R}^r, E_d))$$

We can then describe S as the set of those $x \in V^r(\mathcal{V})$ for which $\{x\} \times M_d \subseteq T$ and consequently S is also open by compactness of M_d. $\qquad\qquad\square$

1.3. Examples and the structure of M_d

We will now give few examples to explain what the spaces M_d look like. We specialize to the case $\mathcal{V} = \mathcal{R} = C^\infty(M, \mathbb{R})$ and write $M_d(M) := M_d(C^\infty(M, \mathbb{R}))$. First we prove a useful lemma which makes defining smooth maps (and in particular immersions) into $M_d(M)$ a bit easier.

LEMMA 1.33. *Let N be a smooth manifold, let $D \pitchfork M_d(M)$ and let*

$$\varphi : \mathcal{R} \times N \to \mathbb{R}^d$$

be a map such that for each $f \in \mathcal{R}$ the partial map $\varphi(f, -) : N \to \mathbb{R}^d$ is smooth and such that for each $x \in N$ the partial map $\varphi(-, x) : \mathcal{R} \to \mathbb{R}^d$ is surjective linear whose kernel $I_x = \ker \varphi(-, x)$ is an ideal in $\mathcal{R}$. Then the map

$$\psi : N \to U_D M_d(M) \subseteq G_d(D)$$

sending $x \in N$ to $U_D(I_x)$ is smooth. If moreover $D \pitchfork M_{d+1}(M)$ then $X \in T_x N$ lies in the kernel of the differential $\psi_* : TN \to TG_d(D)$ if and only if

$$I_x \subseteq \{f \in \mathcal{R} \mid d(\varphi(f, -))(X) = 0\} \tag{1.3}$$

PROOF. Let $x \in N$ and denote $L = \psi(x) = D \cap \ker \varphi(-, x)$. We choose a complementary subspace F to L inside D and get a chart on $G_d(D)$

$$\mathrm{Hom}(L, F) \longrightarrow G_d(D)$$

given by sending a map α to its graph inside $L \times F \cong D$. Let us consider the restriction φ_D of φ to $D \times N$ and write it in the form

$$\varphi_D : L \times F \times N \longrightarrow \mathbb{R}^d$$

Observe that the differential $d\varphi_D|_F$ is an isomorphism of F on $\mathbb{R}^d$ near $L \times F \times \{x\}$. In particular there is a unique solution to the equation

$$\varphi_D(v, \alpha(y)(v), y) = 0$$

and it is automatically smooth. Clearly $\alpha(y) : L \to F$ is the expression of $\psi(y)$ in the above coordinate chart (with $\alpha(x) = 0$). Moreover we have a formula for the derivative

$$d\alpha(X)(v) = (d(\varphi_D(v, -, x)))^{-1}(d(\varphi_D(v, 0, -)))(X)$$

In particular $X \in \ker \psi_*$ if and only if for each $v \in L$ it lies in the kernel of $d(\varphi_D(v, 0, -))$. To explain this condition we introduce

$$I_X := \{f \in \mathcal{R} \mid \varphi(f, x) = 0, \ d(\varphi(f, -))(X) = 0\}$$

As the name suggests it is an ideal and to prove this one observes that for each $y \in N$ we have a multiplication on $\mathbb{R}^d$ arising from the identification $\mathcal{R}/I_y \cong \mathbb{R}^d$. This family is smooth in the sense of the map

$$\mu : N \longrightarrow \mathrm{Hom}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^d)$$

being smooth. If we temporarily denote

$$f(x) := \varphi(f, x) \quad \text{and} \quad df(X) := d(\varphi(f, -))(X)$$

then for $f, g \in \mathcal{R}$ we get

$$d(fg)(X) = \mu(x)\big(f(x) \otimes dg(X) + df(X) \otimes g(x)\big) + d\mu(X)(f(x) \otimes g(x))$$

Therefore if one of f, g lies in I_X then so does their product.

The condition (1.3) from the statement is then equivalent to $I_X = I_x$. Assuming that this equality holds, every $v \in L \subseteq I_x$ lies in I_X implying that $d(\varphi_D(v, 0, -))(X) = 0$. Therefore in this case $\psi_*(X) = 0$. If on the other hand $I_X \subsetneq I_x$ then there is an ideal J which is maximal among those for which $I_X \subseteq J \subsetneq I_x$. Necessarily $J \in \mathrm{M}_{d+1}(M)$ and by our assumption $D \cap J \subsetneq D \cap I_x = L$ so that there is $v \in L$ for which $v \notin I_X$ implying that $d(\varphi_D(v, 0, -))(X) \neq 0$ and $\psi_*(X) \neq 0$. $\qquad\square$

REMARK. The first part of the proof applies even when $\varphi(f, -)$ is merely continuous proving that ψ is also continuous in this situation.

The space $\mathrm{M}_d(M)$ contains as a subspace the configuration space

$$M^{[d]} = M^{(d)}/\Sigma_d \subseteq \mathrm{S}_d$$

We will show now that it is in general an embedded submanifold. I do not know whether $\mathrm{M}_d(M)$ is itself a manifold with corners (or a manifold stratified subset) with the configuration space $M^{[d]}$ being its interior. It is certainly an interesting question to pursue.

PROPOSITION 1.34. *Let D be a finite dimensional linear subspace of $\mathcal{R}$ such that $D \pitchfork M_{d+1}(M)$. Then the inclusion*

$$\psi : M^{[d]} \subseteq S_d \to M_d(M) \subseteq G_d(D)$$
$$Y \mapsto \mathfrak{m}_Y$$

is a smooth embedding.

PROOF. As the map in question has a continuous inverse (restriction of π from Theorem 1.28) we only need to show that it is an immersion. First we express ψ locally via a map $\varphi : \mathcal{R} \times M^{[d]} \to \mathbb{R}^d$ as in Lemma 1.33 and compute the kernel of ψ_* using the same lemma. Therefore let $(x_1, \ldots, x_d) \in M^{(d)}$ and identify a neighbourhood of $[(x_1, \ldots, x_d)] \in M^{[d]}$ with a product $U_1 \times \cdots \times U_d$ of disjoint neighbourhoods U_i of x_i. Then we can define $\varphi : \mathcal{R} \times U_1 \times \cdots \times U_d \to \mathbb{R}^d$ by

$$(f, y_1, \ldots, y_d) \mapsto (f(y_1), \ldots, f(y_d))$$

Clearly all the assumptions of Lemma 1.33 are satisfied and so ψ is a smooth map. Also for

$$(X_1, \ldots, X_d) \in T_{x_1} U_1 \times \cdots \times T_{x_d} U_d$$

we have $d(\varphi(f, -))(X_1, \ldots, X_d) = (df(X_1), \ldots, df(X_d))$ and this can be zero on $\mathfrak{m}_{\{x_1, \ldots, x_d\}}$ only if $X_1 = \cdots = X_d = 0$. $\qquad\square$

Now we investigate a (rather trivial) example where Proposition 1.34 completely describes the topology of $M_d(M)$.

EXAMPLE 1.35 (a description of $M_1(M)$). As ideals of codimension 1 are exactly the maximal ones the canonical map $M \to M_1(M)$ is a bijection. As it was shown to be a smooth embedding in Proposition 1.34 it is a diffeomorphism.

Another easy example is that of the space of ideals for a one-dimensional manifold.

EXAMPLE 1.36 (one-dimensional manifolds). It is clear that the only ideals in $J_0^r(\mathbb{R}, \mathbb{R})$ are the powers of the maximal ideal $\mathfrak{m}_0$. In other words there is exactly one ideal for each codimension and therefore the canonical map

$$M_d(M) \longrightarrow S_d(M)$$

is bijective. As it is also a continuous map between compact Hausdorff spaces it is even a homeomorphism. In particular

$$M_d(M) \cong S_d(M) = M^d / \Sigma_d$$

We will identify this space explicitly for $M = I$, the closed interval. Namely, we claim that $M_d(I)$ is the d-dimensional simplex Δ^d. This is easily seen from the classical subdivision of I^d into simplices

$$e_\sigma = \{(t_1, \ldots, t_d) \mid 0 \le t_{\sigma(1)} \le \cdots \le t_{\sigma(d)} \le 1\}$$

indexed by permutations $\sigma \in \Sigma_d$. Note that the action of Σ_d just permutes them $\tau : e_\sigma \to e_{\tau\sigma}$ (if we think of it as the left action) and therefore the composition

$$\Delta^d \xrightarrow{\;e_{\mathrm{id}}\;} I^d \longrightarrow I^d / \Sigma_d$$

is a continuous bijection between compact Hausdorff spaces hence a homeomorphism[2].

As S^1 is a quotient of I one can conclude that also $S_d(S^1)$ is a quotient of $S_d(I)$. More precisely there is a map $f : \partial_0 \Delta^d \to \partial_d \Delta^d$ from the 0-th face of Δ^d to its d-th face (an affine map sending the i-th vertex e_i of Δ^d to e_{i-1}) such that $S_d(S^1) \cong \Delta^d/f$, the space obtained from Δ^d by identifying the two faces via f (e.g. $S_2(S^1)$ is the Möbius strip). Also note that locally $M_d(S^1)$ looks like $M_d(I)$ and so it is a manifold with corners. $\qquad\square$

In order to describe $M_2(M)$ for a general manifold M we first consider the following construction, a slight modification of a blow-up construction (cf. exercises 7-8 after Chapter 12 in [**BJ**]):

CONSTRUCTION 1.37. Let $E \to N$ be a smooth vector bundle and let us choose an inner product on E. The construction below does not depend on this choice and in fact can be given in an inner product free way. Consider the unit sphere bundle $SE \to N$ and a trivial ray bundle $\eta E = \mathbb{R}_+ \times SE \to SE$ over SE. We have a canonical map $\eta E \to E$ given by sending (t, v) to tv. Clearly it is a diffeomorphism away from the zero sections: $\eta E - 0 \xrightarrow{\cong} E - 0$. One can also think of ηE as E with an open disk subbundle removed but then there is no preferred way of defining a projection map $\eta E \to E$ (the inclusion is not what we are after).

Let M be a manifold and N a closed neat submanifold. We consider the normal bundle ν of N and give $M_N := (M - N) \sqcup S\nu$ a structure of a manifold (with corners in general) defined in terms of an embedding $\iota : \nu \hookrightarrow M$ as

$$(M - N) \cup_{\nu - 0} \eta \nu$$

glued along $\nu - 0$ via the embeddings $\iota : \nu - 0 \hookrightarrow M - N$ and $\nu - 0 \cong \eta\nu - 0 \hookrightarrow \eta\nu$. It can be shown that the resulting manifold does not depend on the choice of the embedding ι.

Now let us specialize to a manifold M equipped with an involution $\tau : M \to M$ and denote the fixed point submanifold by $N = M^\tau$. As $M - N$ is clearly dense in M_N there may be only one possible smooth extension of the involution $\tau : M \to M$ to M_N. We show that it exists and that on $S\nu$ it is just multiplication by -1.

To do so we choose a τ-invariant Riemannian metric on M. The action of τ on $TM|_N$ is by an orthogonal involution and therefore this bundle decomposes into a direct sum

$$TM|_N = TN \oplus \nu$$

with TN being the $(+1)$-eigenspace and ν the (-1)-eigenspace. The canonical embedding $\iota : \nu \hookrightarrow M$ given by

$$(x, v) \mapsto \exp_x(v)$$

transforms the involution τ to multiplication by -1 (as τ is an isometry and therefore preserves geodesics) and this can be clearly extended to $\eta\nu$. As the extended involution $\hat\tau : M_N \to M_N$ has no fixed points $M_{\hat\tau} := M_N/\hat\tau$ is again a manifold with corners.

---

[2]Using Lemma 1.33 on the map $\varphi : \mathcal{R} \times \Delta^d \to \mathbb{R}^d$ defined by

$$(f, (t_1, \ldots, t_d)) \mapsto (\mathcal{I}(f, [t_1]), \ldots, \mathcal{I}(f, [t_1, \ldots, t_d]))$$

one can even produce a diffeomorphism $\Delta^d \to M_d(I)$.

As a set it is a disjoint union of $(M - M^\tau)/\tau$ with the projective bundle $S\nu/\{\pm 1\}$ of the fixed point submanifold M^τ in M. In other words one could say that it is a "blow up" of M^τ in M/τ (but as M/τ is not a manifold this differs from the classical construction and in particular one introduces a new boundary as was mentioned).

EXAMPLE 1.38 (a description of $M_2(M)$). Another easy example is that of codimension 2 ideals. We claim that for the involution $\tau : M^2 \to M^2$ switching the two coordinates, the above constructed $(M^2)_{\hat\tau}$ is homeomorphic to $M_2(M)$.

As $(M^2)_{\hat\tau}$ is easily seen to be compact, we only need to produce a continuous map $(M^2)_{\hat\tau} \to M_2(M)$ and show that it is bijective. By the definition of $(M^2)_{\hat\tau}$, such a map can be obtained from two $\hat\tau$-invariant maps

$$M^2 - \Delta \longrightarrow M_2(M) \qquad \text{and} \qquad \eta\nu \longrightarrow M_2(M)$$

where ν is the normal bundle of the diagonal Δ in M^2 and these maps have to agree on their "common domain". The canonical map $(x, y) \mapsto \mathfrak{m}_{\{x,y\}}$ will certainly do for the first (see Proposition 1.34). Choose a Riemannian metric on M. It is well-known that $\nu \cong TM$ and that the inclusion $\nu \cong TM \hookrightarrow M^2$ can be chosen to be $(x, v) \mapsto (x, \exp_x v)$ where $\exp$ is defined via geodesics[3]. We define

$$\varphi : \eta\nu \times \mathcal{R} = \mathbb{R}_+ \times STM \times \mathcal{R} \to \mathbb{R}^2$$
$$(t, (x, v), f) \mapsto \left(f(x), \frac{f(\exp_x(tv)) - f(x)}{t} \right)$$

where for $t = 0$ the second coordinate is to be interpreted as a limit for $t \to 0$, i.e. as $df(x)(v)$. As φ satisfies all the hypotheses of Lemma 1.33 we conclude that the corresponding map $\psi : \eta\nu \to M_2(M)$ is smooth (in fact an immersion by an easy computation). The two maps clearly agree on their common domain and thus define a map

$$(M^2)_\Delta \longrightarrow M_2(M)$$

Also it is clear that this map is invariant under the involution $\hat\tau$ and finally induces a map $(M^2)_{\hat\tau} \to M_2(M)$. From the set theoretic description of $(M^2)_{\hat\tau}$ as a disjoint union $M^{[2]} \sqcup (STM/\{\pm 1\})$ it is easily seen to be bijective: the ideals $I \in M_2(M)$ have either 2 points in their spectrum and then they lie in $M^{[2]}$, or only 1 point, say x, in which case $(\mathfrak{m}_x)^2 \subseteq I \subseteq \mathfrak{m}_x$ and therefore I can be thought of as a hyperplane in $\mathfrak{m}_x/(\mathfrak{m}_x)^2 \cong (T_x M)^*$. These correspond to one-dimensional subspaces of $T_x M$, i.e. elements of $(STM)_x/\{\pm 1\}$.

As both spaces are compact Hausdorff this map is a homeomorphism (in fact a diffeomorphism). Again note that $M_2(M)$ is always a manifold with corners. $\square$

EXAMPLE 1.39 (a description of $M_3(S^m)$). Let us first classify all ideals

$$I \subseteq J_0^2(\mathbb{R}^m, \mathbb{R})$$

of codimension 3. We claim that there exist only the following two types of such ideals. The ideals of the first type are determined by an orbit of 1-jets of immersions

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[3]More precisely $t \mapsto \exp_x(tv)$ is the unique geodesic starting at x with speed v.

$\sigma : (\mathbb{R}^2, 0) \hookrightarrow (\mathbb{R}^m, 0)$ under the action of the origin preserving diffeomorphism group on $\mathbb{R}^2$. The corresponding ideal is

$$I_{[j_0^1 \sigma]} = \{j_0^2 f \mid j_0^1(f\sigma) = 0\}$$

It can be easily seen that these ideals are all distinct. Similarly the second type is determined by an orbit of 2-jets of immersions $\gamma : (\mathbb{R}, 0) \hookrightarrow (\mathbb{R}^m, 0)$ and the corresponding ideal is

$$I_{[j_0^2 \gamma]} = \{j_0^2 f \mid j_0^2(f\gamma) = 0\}$$

Again all these ideals are distinct.

We prove the claim by reduction to ideals in $J_0^1(\mathbb{R}^m, \mathbb{R})$. Denoting the maximal ideal of $\mathcal{J} := J_0^2(\mathbb{R}^m, \mathbb{R})$ by $\mathfrak{m}$ we identify $J_0^1(\mathbb{R}^m, \mathbb{R})$ with $\mathcal{J}/\mathfrak{m}^2$. Therefore if I is an ideal of $\mathcal{J}$ of codimension 3 its image in $J_0^1(\mathbb{R}^m, \mathbb{R})$ is $(I + \mathfrak{m}^2)/\mathfrak{m}^2$ and its codimension must be 2 or 3 by an easy inspection. As we saw at the end of the last example for codimension 2 in suitable linear coordinates we can write

$$I + \mathfrak{m}^2 = \langle x_2, \ldots, x_m, x_1^2 \rangle$$

Therefore $x_2 + q_2, \ldots, x_m + q_m \in I$ where $q_2, \ldots, q_m$ are some quadratic functions. Necessarily

$$\langle x_2, \ldots, x_m \rangle \mathfrak{m} \subseteq I$$

and therefore we can assume $q_i = c_i \cdot x_1^2$. By another linear change of coordinates we can achieve

$$\langle x_2 + c \cdot x_1^2, x_3, \ldots, x_m \rangle \subseteq I$$

As the codimensions are equal we must have an equality and the left hand side is the ideal $I_{[j_0^2 \gamma]}$ for $\gamma(t) = (t, -c \cdot t^2, 0, \ldots, 0)$.

If the codimension of $I + \mathfrak{m}^2$ is 3 then for dimensional reasons $I = I + \mathfrak{m}^2$ and analogously to the previous case we have

$$I = \langle x_3, \ldots, x_m, x_1^2, x_1 x_2, x_2^2 \rangle$$

in suitable linear coordinates. For $\sigma(s, t) = (s, t, 0, \ldots, 0)$ we get $I = I_{[j_0^1 \sigma]}$.

Now we will explain how $M_3(S^m)$ is identified with the space

$$X := \{(p, x_1, x_2, x_3) \mid p \text{ an affine plane in } \mathbb{R}^{m+1}; x_1, x_2, x_3 \in p \cap S^m\}/\Sigma_3$$

where Σ_3 acts by permuting x_1, x_2, x_3. First we construct a map $\psi : X \to M_3(S^m)$ by specifying it on various subsets:

- When p is tangent to S^m then necessarily $x_1 = x_2 = x_3$ and p, being a 2-dimensional subspace of $T_{x_1} S^m$, can be thought of as an orbit of 1-jets of immersions $(\mathbb{R}^2, 0) \to (S^m, x_1)$. We assign to the class of (p, x_1, x_2, x_3) the corresponding ideal $I_{[j_0^1 \sigma]}$.
- If $x_1 = x_2 = x_3$ but p is not tangent to S^m then it intersects S^m in a circle which can be thought of as an orbit of 2-jets of immersions $(\mathbb{R}^1, 0) \to (S^m, x_1)$. Again the class is sent to the corresponding $I_{[j_0^2 \gamma]}$.
- When $x_1 = x_2 \neq x_3$ then p prescribes a tangential line l at x_1 and the class is sent to $I_l \cdot \mathfrak{m}_{x_3}$ where I_l is the ideal of codimension 2 corresponding to $l \in (STM)_{x_1}/\{\pm 1\}$ as in the previous example.
- If all x_1, x_2, x_3 are distinct then the image will be taken to be $\mathfrak{m}_{\{x_1, x_2, x_3\}}$.

Both X and $\mathrm{M}_3(S^m)$ are compact. We postpone the proof of continuity of ψ and show that it is bijective. If all x_1, x_2, x_3 are distinct then the plane p is determined by these three points as well as it is by a point and a tangential direction at a different point. This exhausts all the ideals of codimension 3 with spectrum consisting of more than one point. On them ψ is a bijection. We classified all the ideals of codimension 3 with spectrum a singleton above. On ideals of type $I_{[j_0^1 \sigma]}$ we clearly get a bijection too. The same is true of ideals of type $I_{[j_0^2 \gamma]}$ when one observes that orbits of 2-jets of immersions $(\mathbb{R}, 0) \hookrightarrow (S^m, x)$ are in bijection with circles through x.[4]

It remains to prove continuity of ψ. We use a continuous version of Lemma 1.33. First we need to resolve X by a bigger space

$$Y := \{(z, u, v, y_1, y_2, y_3, w_{12}, w_{23}, w_{31}) \mid z \in D^{m+1}; u, v \in S^m; u \perp z, v \perp z, u \perp v;$$
$$y_i \in S^1; w_{ij} \in S^1; y_i - y_j \in \mathbb{R}_+ w_{ij}; y_i = y_j \Rightarrow w_{ij} \perp y_i; \text{not all } w_{ij} \text{ the same}\}$$

Here z, u, v prescribe an affine plane p in $\mathbb{R}^{m+1}$ with orthonormal basis (endowing p with an orientation) that clearly intersects S^m. We denote by $\alpha : \mathbb{R}^2 \to \mathbb{R}^{m+1}$ the affine map (isometry) sending 0 to z and the standard basis vectors to u and v. Clearly α depends smoothly on z, u, v. Next we set $\rho = \sqrt{1 - |z|^2}$, the radius of the circle $p \cap S^m$. If p is not tangent to S^m then y_1, y_2, y_3 can be identified with points $x_i = \alpha(\rho \cdot y_i)$ in $p \cap S^m$. Otherwise they are thought of as directions in the tangent space (and clearly $x_i = z$). When y_i are distinct then $w_{ij} = \frac{y_i - y_j}{|y_i - y_j|}$ and if $\rho > 0$ then also $\alpha(w_{ij}) = \frac{x_i - x_j}{|x_i - x_j|}$. If $y_i = y_j$ the vector w_{ij} prescribes an infinitesimal direction from y_j to y_i. The only condition is that when all three points collapse one of these infinitesimal directions has to be opposite to the other two.

Clearly X is a quotient space of Y and it is enough to prove continuity of the map $\tilde{\psi} : Y \to \mathrm{M}_3(S^m)$. On the open subset

$$Z := \{(z, u, v, y_1, y_2, y_3, w_{12}, w_{23}, w_{31}) \in Y \mid |z| < 1; y_1, y_2, y_3 \text{ all distinct}\}$$

we have a continuous map $\varphi : \mathcal{R} \times Z \to \mathbb{R}^3$ defined by sending (f, P) to

$$\left(f(x_1), \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}, \rho \cdot \frac{|f(x_1)(x_2 - x_3) + f(x_2)(x_3 - x_1) + f(x_3)(x_1 - x_2)|}{|x_2 - x_3| \cdot |x_3 - x_1| \cdot |x_1 - x_2|} \right)$$

To show how φ continuously extends to Y we use our version of Taylor expansion (Corollary 1.20) on some extension of f to $\mathbb{R}^{m+1}$.

$$f(x_2) = f(x_1) + \mathcal{I}(f, [x_1, x_2])(x_2 - x_1)$$
$$f(x_3) = f(x_1) + \mathcal{I}(f, [x_1, x_2])(x_3 - x_1) + 1/2 \cdot \mathcal{I}(f, [x_1, x_2, x_3])(x_3 - x_1, x_3 - x_2)$$

and for brevity we put $a = \mathcal{I}(f, [x_1, x_2]), b = \mathcal{I}(f, [x_1, x_2, x_3])$. Then easily

$$\frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} = |a(\alpha(w_{12}))|$$

---

[4]This is clear from the following two observations. Firstly a 2-jet of a path $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^{m+1}, x)$ is a 2-jet of $(\mathbb{R}, 0) \to (S^m, x)$ iff $j_0^2(\langle \gamma, \gamma \rangle)$ is constant 1. For any parametrization of a circle $c \subseteq \mathbb{R}^{m+1}$ this condition is equivalent to $c \subseteq S^m$ by an easy computation. Applying these considerations to the osculation circle of a path $\gamma : (\mathbb{R}, 0) \to S^m$ shows that it always lies on S^m providing the bijection.

$$f(x_1)(x_2 - x_3) + f(x_2)(x_3 - x_1) + f(x_3)(x_1 - x_2)$$
$$= a(x_1 - x_2)(x_2 - x_3) - a(x_2 - x_3)(x_1 - x_2) - 1/2 \cdot b(x_3 - x_1, x_2 - x_3)(x_1 - x_2)$$

Now if we denote $a^* = a(u)u + a(v)v$ the projection of the gradient of a into the plane spanned by u and v we can rewrite

$$a(x_1 - x_2)(x_2 - x_3) - a(x_2 - x_3)(x_1 - x_2)$$
$$= \langle a^*, x_1 - x_2 \rangle (x_2 - x_3) - \langle a^*, x_2 - x_3 \rangle (x_1 - x_2)$$
$$= \mathrm{vol}(x_1 - x_2, x_2 - x_3)Ra^*$$

where vol denotes the oriented volume and R rotation in p by $+\pi/2$ so that $Ra^* = a(u)v - a(v)u$. Easily $\mathrm{vol}(x_1 - x_2, x_2 - x_3)$ is twice the area of the triangle $x_1x_2x_3$. It is well known that this can be expressed by the lengths of the sides and the radius of the circumscribed circle as

$$a(x_1 - x_2)(x_2 - x_3) - a(x_2 - x_3)(x_1 - x_2) = \mathrm{sign} \cdot 1/(2\rho) \cdot |x_1 - x_2||x_2 - x_3||x_3 - x_1|Ra^*$$

Here sign depends on the orientation of the triangle $x_1x_2x_3$ inside p which on the other hand depends continuously on the data[5]. Therefore the last component of φ is

$$1/2 \cdot |\mathrm{sign} \cdot Ra^* - \rho \cdot b(\alpha(w_{31}), \alpha(w_{23}))\alpha(w_{12})|$$

These formulas obviously extend continuously to Y and Lemma 1.33 then provides a continuous map

$$\tilde{\psi} : Y \to \mathrm{M}_3(S^m)$$

It is easy to verify that ψ factors through the quotient X of Y yielding the above described map ψ and showing that it is indeed a homeomorphism.

EXAMPLE 1.40 (description of $\mathrm{M}_1(\mathcal{V})$ for a special $\mathcal{V}$). Let $N \subseteq M$ be a closed neat[6] submanifold and

$$\mathcal{V} = \{f \in C^\infty(M, \mathbb{R}) \mid f(z) = 0 \; \forall z \in N\}$$

We show later in Example 1.43 that $\mathcal{V}$ is topologically finitely generated. We can argue similarly as in the last example to conclude that

$$\mathrm{M}_1(\mathcal{V}) \cong (M - N) \sqcup (S\nu/\{\pm 1\})$$

where the right hand side is given a smooth structure in a way similar to Construction 1.37 but using the projective bundle with its canonical line bundle over it in place of the sphere bundle. This construction is called a (real) blow-up (cf. exercises 7-8 after Chapter 12 in [**BJ**]).

Our next goal is to describe certain subsets of $\mathrm{M}_d(M)$. They are subsets of ideals of a "fixed type" and are injectively immersed submanifolds. Also every ideal has some (unique) type and so $\mathrm{M}_d(M)$ is in fact a disjoint union (over all possible types) of these submanifolds.

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[5]Easily Y decomposes into a disjoint union of two subspaces according to the orientation of $y_1y_2y_3$ in $\mathbb{R}^2$ (this is clear for nondegenerate triangles and Y is designed so as to remember via w_{12}, w_{23}, w_{31} the orientation of degenerate triangles). This decomposition prescribes a continuous function $\mathrm{sign} : Y \to \mathbb{Z}/2 = \mathbb{Z}^*$

[6]see [**Hir**]

We take the following construction from section 35 of [**KMS**]. A *Weil algebra* is a finite dimensional associative, commutative algebra A over $\mathbb{R}$ with a unit such that $A = \mathbb{R} \oplus N$ where N is the ideal of nilpotent elements. Equivalently it could be described as a quotient algebra of $J_0^r(\mathbb{R}^m, \mathbb{R})$ for some r and m. Therefore we can get all ideals of $\mathcal{R}$ with a spectrum consisting of a single point as kernels of surjective algebra homomorphisms $\mathcal{R} \to A$ for some Weil algebra A (namely A is the quotient of $\mathcal{R}$ by that ideal). We give the set of all such homomorphisms (surjective or not) a smooth structure in such a way that the map sending such a homomorphism to the spectrum of its kernel is a bundle projection. This bundle is called the *Weil bundle* associated to A.

We first give a construction of this bundle and then show that its points can be indeed identified with homomorphism $\mathcal{R} \to A$. We start with the restriction

$$J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M) \to M \tag{1.4}$$

of the jet bundle $J^r(\mathbb{R}^m, M) \to M$ to the subspace of all invertible jets with source 0. Setting $G_m^r := J_{0,\mathrm{diff}}^r(\mathbb{R}^m, \mathbb{R}^m)_0$, the Lie group of all invertible jets with source and target 0, we see that (1.4) is a principal G_m^r-bundle and so we can define

$$T_A M := J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M) \times_{G_m^r} \mathrm{Hom}_{\mathrm{alg}}(J_0^r(\mathbb{R}^m, \mathbb{R}), A) \tag{1.5}$$

This clearly expresses $T_A M$ as a smooth bundle over M with fibre

$$\mathrm{Hom}_{\mathrm{alg}}(J_0^r(\mathbb{R}^m, \mathbb{R}), A) \cong N^m$$

Moreover we have a bijection defined in terms of (1.5) by the formula

$$T_A M \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{alg}}(\mathcal{R}, A)$$
$$[j_x^r g, \varphi] \mapsto \left(\mathcal{R} \xrightarrow{j_x^r} J_x^r(M, \mathbb{R}) \xrightarrow{g^*} J_0^r(\mathbb{R}^m, \mathbb{R}) \xrightarrow{\varphi} A \right)$$

It is easily seen to be a bijective correspondence (that a kernel of any $\mathcal{R} \to A$ has a spectrum consisting of only a single point follows from the fact that in A, the only idempotents are 0 and 1). We have a subbundle

$$\check{T}_A M := J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M) \times_{G_m^r} \mathrm{SurHom}_{\mathrm{alg}}(J_0^r(\mathbb{R}^m, \mathbb{R}), A)$$

which then corresponds to surjective algebra homomorphisms $\mathcal{R} \to A$.

One says that an ideal I is of type A if $\mathcal{R}/I \cong A$. An ideal I of type A can then be identified with a class of surjective homomorphisms $\mathcal{R} \to A$, namely with the class of all those homomorphisms that have kernel I. In this way we get a space $J^A M$ of all ideals of type A as a certain quotient of $\check{T}_A M$. A crucial observation in [**Alo**] is that the action of G_m^r on $\mathrm{SurHom}_{\mathrm{alg}}(J_0^r(\mathbb{R}^m, \mathbb{R}), A)$ is transitive and so, after choosing some $\alpha_0 \in \mathrm{SurHom}_{\mathrm{alg}}(J_0^r(\mathbb{R}^m, \mathbb{R}), A)$, one can identify it with the quotient of G_m^r by the stabilizer of α_0. In the same terminology an ideal in $J_0^r(\mathbb{R}^m, \mathbb{R})$ of type A is a class of G_m^r modulo the stabilizer of $\ker \alpha_0$. Therefore the space of ideals of type A can be identified with the smooth bundle

$$J^A M \cong J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M)/\mathrm{St}(\ker \alpha_0) \longrightarrow M$$

and clearly the smooth structure does not depend on the choice of α_0.

PROPOSITION 1.41. *Let $D \pitchfork \mathrm{M}_{d+1}(M)$ be a finite dimensional linear subspace of $\mathcal{R}$ and let A be a d-dimensional Weil algebra. Then the inclusion*

$$\iota : J^A M \subseteq \mathrm{M}_d(M) \subseteq G_d(D)$$

is an injective immersion.

PROOF. This is another application of Lemma 1.33. Consider the map

$$\varphi : \mathcal{R} \times J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M) \to A$$
$$(f, j^r_x(g)) = \alpha_0(j^r_x(fg))$$

Clearly this map is surjective linear in the first and smooth in the second variable and hence in the sense of Lemma 1.33 it defines

$$\psi : J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M) \to \mathrm{M}_d(M) \subseteq G_d(D)$$

which is also smooth. As we have a commutative diagram

$$
\begin{array}{ccc}
J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M) & \xrightarrow{\ \ \psi\ \ } & \mathrm{M}_d(M) \subseteq G_d(D) \\
\downarrow & \nearrow & \\
J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M)/\mathrm{St}(\ker \alpha_0) & \ \ \iota\ \ &
\end{array}
$$

in order to show that the dashed arrow is an immersion we need to identify $\ker \psi_*$. Lemma 1.33 gives an answer in terms of the kernel of the differential (say at $j^r_x g$) of the map $\varphi(f, -)$ which can be decomposed as

$$J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M) \xrightarrow{f_*} J^r_0(\mathbb{R}^m, \mathbb{R}) \xrightarrow{\alpha_0} A$$

To give a tangent vector in $T_{j^r_0 g} J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M)$ is the same as to give an element of $T_{\mathrm{id}} J^r_{0,\mathrm{diff}}(\mathbb{R}^m, \mathbb{R}^m)$ and then compose with g. The elements of $T_{\mathrm{id}} J^r_{0,\mathrm{diff}}(\mathbb{R}^m, \mathbb{R}^m)$ arise from vector fields. Therefore let $X : \mathbb{R}^m \to T\mathbb{R}^m$ be a local vector field with a local flow

$$\gamma : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^m$$

Under our identifications it defines a tangent vector

$$\hat{X} := \frac{d}{dt}\bigg|_{t=0} j^r_0(g\gamma(-, t)) \in T_{j^r_0 g} J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M)$$

Then for each $f \in \mathcal{R}$ we get

$$d(\alpha_0 f_*)(\hat{X}) = \frac{d}{dt}\bigg|_{t=0} \alpha_0(j^r_0(fg\gamma(-, t))) = \alpha_0(j^r_0(X(fg)))$$

Suppose that $X(0) \neq 0$. Then we claim that there exists an $f \in \psi(j^r_x g)$ for which this expression is nonzero as well. In other words such $\hat{X}$ can never lie in $\ker \psi_*$. We postpone the proof of this claim and thus assume that the only $\hat{X}$ which could produce an element in this kernel are the vectors tangent to the submanifold $J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M)_x$. The Lie group G^r_m acts simply transitively on this space. Let Y be an element of the Lie algebra of G^r_m. Then we obtain a vector field (with p running over $J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M)_x$)

$$Y^+(p) := \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tY)$$

and we also have similar vector fields on $J_0^r(\mathbb{R}^m, \mathbb{R})$. The restriction of $\varphi(f, -)$ is simply the composition

$$J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M)_x \xrightarrow{f_*} J_0^r(\mathbb{R}^m, \mathbb{R}) \xrightarrow{projection} J_0^r(\mathbb{R}^m, \mathbb{R})/\ker\alpha_0 \cong A$$

Therefore $Y^+(j_0^r g) \in \ker\psi_*$ if and only if for each $f \in \psi(j_0^r g)$ we have $df_*(Y^+(j_0^r g)) \in \ker\alpha_0$. As the map f_* is G_m^r-equivariant we can rewrite

$$df_*(Y^+(j_0^r g)) = Y^+(f_* j_0^r g) = Y^+(j_0^r(fg))$$

for the corresponding canonical vector field on $J_0^r(\mathbb{R}^m, \mathbb{R})$. Now observe that we get all possible values $j_0^r(fg) \in \ker\alpha_0$ by varying f over $\psi(j_0^r g)$. Therefore $Y^+(j_0^r g) \in \ker\psi_*$ if and only if $Y^+(\ker\alpha_0) \subseteq \ker\alpha_0$. These Y clearly constitute the Lie algebra of the stabilizer $\mathrm{St}(\ker\alpha_0)$ and therefore we conclude that $\ker\psi_*$ is exactly the vertical tangent bundle of

$$J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M) \longrightarrow J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M)/\mathrm{St}(\ker\alpha_0) \cong J^A M$$

Consequently ψ induces on the quotient $J^A M \cong J_{0,\mathrm{diff}}^r(\mathbb{R}^m, M)/\mathrm{St}(\ker\varphi_0)$ an immersion $\iota : J^A M \longrightarrow G_d(D)$.

Now we prove the remaining claim. Because we assume that $X(0) \neq 0$ we can find a local diffeomorphism $h : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that $h^* X = \partial_{x_1}$. Then

$$X(fg) = \partial_{x_1}(fgh) \circ h^{-1}$$

We denote by K the kernel of

$$J_0^r(\mathbb{R}^m, \mathbb{R}) \xrightarrow{(h^{-1})^*} J_0^r(\mathbb{R}^m, \mathbb{R}) \xrightarrow{\varphi_0} A$$

and we are looking for $f \in \mathcal{R}$ such that $j_0^r(fgh) \in K$ but $j_0^r(\partial_{x_1}(fgh)) \notin K$. Let $K \subseteq (x_1^k, x_2, \ldots, x_m)$ but $K \not\subseteq (x_1^{k+1}, x_2, \ldots, x_m)$. Both g and h being diffeomorphisms there exists $f \in \mathcal{R}$ such that $j_0^r(fgh) \in K - (x_1^{k+1}, x_2, \ldots, x_m)$. Then $fgh = x_1^k \cdot \lambda$ modulo $(x_2, \ldots, x_m)$ with $\lambda(0) \neq 0$ and it is easy to see that $\partial_{x_1}(fgh) = x_1^{k-1}\mu$ modulo $(x_2, \ldots, x_m)$ with $\mu(0) \neq 0$ so that $j_0^r(\partial_{x_1}(fgh))$ does not lie in $(x_1^k, x_2, \ldots, x_m)$ and in particular it does not lie in K. $\qquad\square$

QUESTION. Is the inclusion $\iota : J^A M \hookrightarrow \mathrm{M}_d(M)$ an embedding? Quite easily (reducing to a local question and using polynomials) one can reduce this problem to the question of the canonical map

$$G_m^r/(G_m^r \cap \mathrm{St}(\ker\varphi_0)) \hookrightarrow Gl(J_0^r(\mathbb{R}^m, \mathbb{R}))/\mathrm{St}(\ker\varphi_0)$$

being an embedding.

An easy generalization to the case of finitely many Weil algebras A_i, $i = 1, \ldots, n$, produces a bundle

$$\check{T}_{A_1,\ldots,A_n} M = \left(\check{T}_{A_1} M \times \cdots \times \check{T}_{A_n} M\right)\big|_{M^{(n)}} \longrightarrow M^{(n)}$$

togeter with a bijection $\check{T}_{A_1,\ldots,A_n} M \cong \mathrm{SurHom}_{\mathrm{alg}}(\mathcal{R}, A_1 \times \cdots \times A_n)$. Every ideal $I \in \mathrm{M}_d(M)$ of type $A_1 \times \cdots \times A_n$ can be clearly recovered as a kernel of such surjective homomorphism. In this way we get a space $J^{A_1,\ldots,A_n} M$ of ideals of a fixed

type $A_1 \times \cdots \times A_n$ as a quotient of $\check{T}_{A_1,\ldots,A_n} M$. Moreover we can again identify $J^{A_1,\ldots,A_n} M$ with a quotient[7]

$$(J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M))^{(n)}_{\underline{M}} \times_{G^r_m \wr \Sigma_n} (G^r_m \wr \Sigma_n)/\mathrm{St}(\ker \alpha_0)$$

for some (any) surjective homomorphism $\alpha_0 : (J^r_0(\mathbb{R}^m, \mathbb{R}))^n \to A_1 \times \cdots \times A_n$. The canonical inclusion map $\iota : J^{A_1,\ldots,A_n} M \hookrightarrow \mathrm{M}_d(M)$ with $d = \dim A_1 + \cdots + \dim A_n$ is an injective immersion. For a proof observe that locally $(J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M))^{(n)}_{\underline{M}}$ is just a product of $J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M)$ and so one can almost copy the proof of Proposition 1.41 (also see the proof of Proposition 1.34).

1.4. A criterion for topological finite generation

In this section we give a criterion for $\mathcal{V}$ to be a topologically finitely generated $\mathcal{R}$-module in terms of its local structure allowing us to prove that some interesting topological $\mathcal{R}$-modules are topologically finitely generated.

First we observe that $\mathcal{R}$ is the space of global sections of the sheaf $\mathcal{R}(U) = C^\infty(U, \mathbb{R})$ of (locally convex) topological $\mathbb{R}$-algebras. The sheaf property in the realm of topological spaces is proved in Lemma B.1 whereas the algebraic part is clear. Suppose that $\mathcal{V}(U)$ is a sheaf of topological $\mathcal{R}(U)$-modules and denote $\mathcal{V} = \mathcal{V}(M)$. We have the following result.

PROPOSITION 1.42. *An $\mathcal{R}$-module $\mathcal{V}$ is topologically finitely generated if there is an open covering* U *of M such that for every $U \in$ U, $\mathcal{V}(U)$ is a topologically finitely generated $\mathcal{R}(U)$-module.*

PROOF. First we construct an auxiliary map enabling us to extend sections. Let U be an open subset of M and let $\lambda : M \to \mathbb{R}$ be a smooth function with $\mathrm{supp}(\lambda) \subseteq U$. We define a λ-extension map $e_\lambda : \mathcal{V}(U) \to \mathcal{V}$ which is simply multiplication by λ and extending by 0. To be more precise let $V = M - \mathrm{supp}(\lambda)$. By the sheaf property we have a pullback diagram

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\;res\;} & \mathcal{V}(U) \\
{\scriptstyle res}\downarrow & & \downarrow{\scriptstyle res} \\
\mathcal{V}(V) & \xrightarrow{\;res\;} & \mathcal{V}(U \cap V)
\end{array}
$$

Now we can define $e_\lambda : \mathcal{V}(U) \to \mathcal{V}$ by

$$(l_\lambda, 0)^T : \mathcal{V}(U) \longrightarrow \mathcal{V}(U) \times_{\mathcal{V}(U \cap V)} \mathcal{V}(V) \cong \mathcal{V}$$

the first component being multiplication by λ. By the universal property of pullbacks the composition $\mathcal{V} \xrightarrow{\;res\;} \mathcal{V}(U) \xrightarrow{\;e_\lambda\;} \mathcal{V}$ is multiplication by λ as is the other composition $res \circ e_\lambda$.

---

[7]Here $(J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M))^{(n)}_{\underline{M}}$ denotes the restriction of the power $(J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M))^n \to M^n$ of the jet bundle to the subspace $M^{(n)} \subseteq M^n$. In particular as the original bundle was a principal G^r_m-bundle the resulting bundle over $M^{(n)}$ will be a principal $(G^r_m)^n$-bundle on which there is an action of the symmetric group Σ_n. Taking the quotient by this action one gets a principal $(G^r_m \wr \Sigma_n)$-bundle $(J^r_{0,\mathrm{diff}}(\mathbb{R}^m, M))^{(n)}_{\underline{M}} \to M^{[n]}$.

As an application we prove that if A is a closed subset of M such that $A \subseteq U$ then $\mathcal{R}/\mathfrak{n}_A \cong \mathcal{R}(U)/\mathfrak{n}_A$. A map in one direction is induced by the restriction map and in the other direction by e_λ where $\lambda : M \to \mathbb{R}$ is any smooth map supported in U such that $\lambda = 1$ on a neighbourhood of A.

Now we proceed with the actual proof. Let $\mathrm{U} = \{U_1, \ldots, U_l\}$ be any open covering of M and let $\lambda_1, \ldots, \lambda_l$ be a subordinate partition of unity. We define a gluing map

$$(e_{\lambda_1}, \ldots, e_{\lambda_l}) : \mathcal{V}(U_1) \times \cdots \times \mathcal{V}(U_l) \twoheadrightarrow \mathcal{V}$$

which is a quotient map as it has a section $(res, \ldots, res)^T : \mathcal{V} \to \mathcal{V}(U_1) \times \cdots \times \mathcal{V}(U_l)$. Suppose now that for each i there is a topological quotient map

$$\varphi_i : \mathcal{R}(U_i)E_i \twoheadrightarrow \mathcal{V}(U_i)$$

with E_i a finite dimensional real vector space. Denoting by ψ_i the composition

$$\mathcal{R}E_i \xrightarrow{res} \mathcal{R}(U_i)E_i \xrightarrow{\varphi_i} \mathcal{V}(U_i) \xrightarrow{e_{\lambda_i}} \mathcal{V}$$

we claim that the map $(\psi_1, \ldots, \psi_l) : \mathcal{R}E_1 \times \cdots \times \mathcal{R}E_l \longrightarrow \mathcal{V}$ is also a topological quotient map. This follows from the diagram

$$
\begin{array}{ccccc}
\prod \mathcal{R}E_i & \xrightarrow{\prod res} & \prod \mathcal{R}(U_i)E_i & \xrightarrow{\prod \varphi_i} & \prod \mathcal{V}(U_i) \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle (e_{\lambda_1}, \ldots, e_{\lambda_l})} \\
\prod (\mathcal{R}/\mathfrak{n}_{A_i})E_i & \xrightarrow{\cong} & \prod (\mathcal{R}(U_i)/\mathfrak{n}_{A_i})E_i & \dashrightarrow & \mathcal{V}
\end{array}
$$

where $A_i = \mathrm{supp}(\lambda_i) \subseteq U_i$. Here the dashed arrow exists and is clearly a quotient map. As $\mathcal{R}E_1 \times \cdots \times \mathcal{R}E_l \cong \mathcal{R}(E_1 \times \cdots \times E_l)$ this finishes the proof. $\qquad \square$

EXAMPLE 1.43. Let M be a compact manifold, let $N \subseteq M$ be a closed submanifold and suppose that either N is neat or that $N = \partial M$. Let r be a positive integer. We define a submodule $\mathcal{V} \subseteq \mathcal{R}$ as

$$\mathcal{V} = \{f \in C^\infty(M, \mathbb{R}) \mid j_z^{r-1} f = 0 \ \forall z \in N\}$$

Then $\mathcal{V}$ is a topologically finitely generated $\mathcal{R}$-module.

PROOF. We will give the proof in the case $\partial M = \emptyset$ to simplify the notation. Clearly $\mathcal{V}$ is a subsheaf of $\mathcal{R}$ and thus a sheaf itself. We cover M by coordinate charts $\mathbb{R}^m \cong U \subseteq M$ under which either $N = \emptyset$ or

$$N = \mathbb{R}^{m_0} \subseteq \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \cong \mathbb{R}^m$$

We are left to show that each $\mathcal{V}(U)$ is a topologically finitely generated $\mathcal{R}(U)$-module. Using the charts we can assume that $U = \mathbb{R}^m$ and in fact we will be assuming that $M = U = \mathbb{R}^m$ to avoid writing U everywhere.

When $N = \emptyset$ we have $\mathcal{V} = \mathcal{R}$ and there is nothing to prove. Hence let us assume that $N = \mathbb{R}^{m_0}$. We define E to be the space of homogenous polynomials on $\mathbb{R}^m$ of degree r that belong to $\mathcal{V}$, i.e. those polynomials that do not depend on the $\mathbb{R}^{m_0}$-coordinate. This gives us an embedding $E \to \mathcal{V}$ and we define

$$\varphi : \mathcal{R}E \to \mathcal{V}$$

to be the unique $\mathcal{R}$-module map extending this embedding. In other words, if we think of $\mathcal{R}E$ as $C^\infty(\mathbb{R}^m, E)$ then $g : \mathbb{R}^m \to E$ is send by φ to the function $f : z \mapsto (g(z))(z)$. To prove that it is a topological quotient map we will construct a continuous section (which will happen to be linear but will not be an $\mathcal{R}$-module map. In fact there is no $\mathcal{R}$-section).

Let $(x, y) \in \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \cong \mathbb{R}^m$. We define an r-simplex $\sigma_{(x,y)}$ to be

$$\sigma_{(x,y)} = [(x, 0), \dots, (x, 0), (x, y)]$$

and then for $f \in \mathcal{R} = C^\infty(\mathbb{R}^m, \mathbb{R})$ we set

$$\mathcal{J}(f, x, y) := \frac{1}{r!} \cdot \int_{\Delta^r} f^{(r)}_{\mathbb{R}^{m_1}} \sigma_{(x,y)} \in \mathrm{Hom}(S^r \mathbb{R}^{m_1}, \mathbb{R})$$

where $f^{(r)}_{\mathbb{R}^{m_1}}$ is the r-fold derivative of f in the direction $\mathbb{R}^{m_1}$.

If we assume that $f \in \mathcal{V}$ then by Lemma 1.19

$$f(x, y) = \mathcal{J}(f, x, y)(y, \dots, y)$$

Hence if we identify $\mathrm{Hom}(S^r \mathbb{R}^{m_1}, \mathbb{R})$ with the polynomial space E then we get

$$\mathcal{J}(f, x, y) \in E$$

with the property that

$$f(x, y) = \mathcal{J}(f, x, y)(x, y)$$

Clearly each $\mathcal{J}(f, -) : \mathbb{R}^m \to E$ is a smooth function and so we obtain a section $s : \mathcal{V} \to \mathcal{R}E$ of φ by defining

$$s : f \mapsto \mathcal{J}(f, -)$$

The only thing that remains to be checked is the continuity of this map. Note that the same formula defines a map $s : \mathcal{R} \to \mathcal{R}E$ and hence it is enough to prove that this extension is continuous. This is an easy exercise in differential topology: roughly speaking, if the $(r + k)$-jet of f is small then the k-jet of $s(f)$ is small as well. $\qquad \square$

This readily generalizes to sections of a smooth vector bundle $F \to M$:

$$\mathcal{V} = \{f \in \Gamma F \mid j^{r-1}_z f = 0, z \in N\}$$

is a topologically finitely generated $\mathcal{R}$-module. Here the condition $j^{r-1}_z f = 0$ means that f and the zero section $0 : M \to F$ have the same $(r - 1)$-jet at z. Another generalization is possible. If we are given a finite collection N_i of submanifolds (such that either N_i is neat or $N_i = \partial M$) in general position (i.e. the tangent spaces at any intersection point are in general position) and positive integers r_i then the $\mathcal{R}$-module

$$\mathcal{V} = \bigcap_i \{f \in \Gamma F \mid j^{r_i - 1}_z f = 0 \; \forall z \in N_i\}$$

is a topologically finitely generated $\mathcal{R}$-module. We will indicate the necessary changes in the case of two submanifolds (that intersect transversely). We can assume that $F = \mathbb{R} \times M$ and

$$M = \mathbb{R}^m \cong \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$$

and that $N_1 = \mathbb{R}^{m_0} \times \{0\} \times \mathbb{R}^{m_2}$, $N_2 = \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \{0\}$. Take E to be the space of homogenous polynomials on $\mathbb{R}^m$ of degree $r_1 + r_2$ that belong to $\mathcal{V}$. Again we get an $\mathcal{R}$-module map

$$\varphi : \mathcal{R}E \to \mathcal{V}$$

which under the identification $\mathcal{R}E \cong C^\infty(M, E)$ sends $g : M \to E$ to $f : z \mapsto (g(z))(z)$. To construct the section of φ we define for a point $(w, x, y) \in \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \cong \mathbb{R}^m$ an affine map

$$\sigma_{(w,x,y)} : \Delta^{r_1} \times \Delta^{r_2} \cong \{*\} \times \Delta^{r_1} \times \Delta^{r_2} \xrightarrow{w \times [0,...,0,x] \times [0,...,0,y]} \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$$

and we set

$$\mathcal{J}(f, w, x, y) = \int_{\Delta^{r_1} \times \Delta^{r_2}} f^{(r_1),(r_2)}_{\mathbb{R}^{m_1}, \mathbb{R}^{m_2}} \sigma_{(w,x,y)} \in \mathrm{Hom}(S^{r_1}\mathbb{R}^{m_1} \otimes S^{r_2}\mathbb{R}^{m_2}, \mathbb{R})$$

where $f^{(r_1),(r_2)}_{\mathbb{R}^{m_1}, \mathbb{R}^{m_2}}$ denotes the $(r_1 + r_2)$-fold derivative, r_1-times in the direction $\mathbb{R}^{m_1}$ and r_2-times in the direction $\mathbb{R}^{m_2}$. Again we can identify the target space with E and get the desired section.

EXAMPLE 1.44. Let $F \to M$ be a vector bundle and $J^r F \to M$ its jet prolongation. There are two actions of $\mathcal{R}$ on $\Gamma(J^r F)$. The first one does not use the jet structure and is given in terms of $f \in \mathcal{R}$ and $s \in \Gamma(J^r F)$ by the formula $(f \cdot s)(x) = f(x)s(x)$. The second is defined in the following way: let $x \in M$ and let $s(x) = j^r_x g$. Then we set $(f \cdot s)(x) = j^r_x(fg)$. To distinguish the two actions we denote $\Gamma(J^r F)$ with the latter one by $j^r\Gamma(J^r F)$.

We claim that $j^r\Gamma(J^r F)$ is a topologically finitely generated $\mathcal{R}$-module.

PROOF. Locally we can reduce to the case $F = \mathbb{R} \times M$ and $M = \mathbb{R}^m$. Using trivialization $J^r(\mathbb{R}^m, \mathbb{R}) \cong J^r_*(\mathbb{R}^m, \mathbb{R}) \times \mathbb{R}^m$ we have $\Gamma(J^r F) \cong C^\infty(M, J^r_*(\mathbb{R}^m, \mathbb{R}))$ and think of $J^r_*(\mathbb{R}^m, \mathbb{R}) \subseteq \Gamma(J^r F)$ as the subspace of constant maps. Clearly with the first action $\Gamma(J^r F)$ is free on $J^r_*(\mathbb{R}^m, \mathbb{R})$. We will show that the same is true for $j^r\Gamma(J^r F)$. First note that on the kernel of the canonical map $\alpha : \Gamma(J^r F) \to \Gamma(J^{r-1}F)$ the two actions coincide. Denoting by K_r the kernel of $J^r_*(\mathbb{R}^m, \mathbb{R}) \to J^{r-1}_*(\mathbb{R}^m, \mathbb{R})$ we obtain a commutative diagram with φ_r the unique $\mathcal{R}$-homomorphism extending the above mentioned inclusion $J^r_*(\mathbb{R}^m, \mathbb{R}) \subseteq j^r\Gamma(J^r F)$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}K_r & \longrightarrow & \mathcal{R}J^r_*(\mathbb{R}^m, \mathbb{R}) & \longrightarrow & \mathcal{R}J^{r-1}_*(\mathbb{R}^m, \mathbb{R}) & \longrightarrow & 0 \\
& & \psi_r \downarrow \cong & & \varphi_r \downarrow & & \varphi_{r-1} \downarrow & & \\
0 & \longrightarrow & \ker \alpha & \longrightarrow & j^r\Gamma(J^r F) & \xrightarrow{\alpha} & j^{r-1}\Gamma(J^{r-1}F) & \longrightarrow & 0
\end{array}
$$

As the map on the left is the same as for the first action it must be an isomorphism. Using a splitting of the top row and an induction on r it is easy to produce a (continuous!) inverse of φ_r. $\qquad\square$

1.5. Noncompact manifolds

This section is more of an informative one with no complete proofs. The idea is that for a noncompact M everything should go the same except the compactness of M_d has to be replaced by the properness of the canonical map $\mathrm{M}_d \to \mathrm{S}_d$.

Let us try to summarize what possible complications could arise. The first one is that we do not have a characterization of the maximal ideals in $\mathcal{R}$. There are certainly more than just ones of the form $\mathfrak{m}_x$ for $x \in M$, the other correspond to some behaviour "at infinity" (the second part of the next proof shows that such ideals do not contain any proper function; in fact not even a function with a compact spectrum). Nevertheless the additional maximal ideals have infinite codimension:

Proof (based on the proof of lemma 35.8. of [**KMS**]). Suppose that I is a maximal ideal of finite codimension and such that $\mathrm{sp}(I) = \emptyset$. Take any proper function $h \in \mathcal{R}$. As $\mathcal{R}/I$ is finite dimensional, the vectors $h + I, h^2 + I, \dots$ must be linearly dependent and so there is a nonzero finite linear combination

$$g = \sum_{i=1}^{n} a_i h^i \in I$$

One can see easily that g is itself proper as it is a composition

$$g : M \xrightarrow{h} \mathbb{R} \xrightarrow{p} \mathbb{R}$$

with $p = \sum a_i x^i$ a non-constant polynomial, hence proper. In particular $g^{-1}(0)$ is compact. The collection of closed subsets $\mathrm{sp}(f) = f^{-1}(0)$ with f running over elements of I has by assumption an empty intersection. Therefore (remember $g^{-1}(0)$ is compact) there is a finite number of elements $f_1, \dots, f_n \in I$ such that

$$\mathrm{sp}(f_1) \cap \dots \cap \mathrm{sp}(f_n) = \emptyset$$

Then $f_1^2 + \dots + f_n^2 \in I$ is positive on M and is therefore a unit in $\mathcal{R}$, a contradiction to maximality. $\qquad\square$

Therefore we obtain the same results about the structure of submodules of finite codimension. We can even find a finite dimensional transversal (the proofs did not use compactness).

Problems arise in the proof of Proposition 1.16 (as there exist maximal ideals of infinite codimension now. Nevertheless the conclusions – U_D injective and reflecting inclusions – can be proved easily for $D \pitchfork M_{d+1}$ using intersections) and of continuity of the natural map $\pi : U_D M_d \to S_d$ in Theorem 1.28.

CHAPTER 2

Simplex on a space

The purpose of this chapter is to derive a spectral sequence for computing the Čech cohomology with compact supports of the target of a surjective finite-to-one proper map $f : X \to Y$. The idea (taken from [**Vas**], [**Vas2**]) is to replace Y by its "resolution" Rf. It is a space with the same Čech cohomology as Y and with a natural finite filtration such that $R^0 f = X$ and $R^p f - R^{p-1} f$ has an interpretation in terms of $(p + 1)$-tuples of points mapping to a single point. The spectral sequence is then just a spectral sequence associated with this filtration. The resolution Rf is obtained from the following construction.

Let $\mathcal{F}$ denote the category of finite sets and all maps between them. There is a functor $\Delta : \mathcal{F} \to \mathsf{Top}$ sending K to

$$\Delta K = \text{a ``free'' convex hull of } K$$

To be more precise it is a convex hull of K inside the free $\mathbb{R}$-vector space $\mathbb{R}^K$ on K. We think of $\mathcal{F}$ as a (full) subcategory of the category Top of compactly generated Hausdorff spaces. The topological left Kan extension of Δ along the inclusion yields a functor $\mathsf{Top} \to \mathsf{Top}$ which we will also denote by Δ

$$\Delta X := \int^{K \in \mathcal{F}} \mathsf{map}(K, X) \times \Delta K$$

(for coends see Section IX.6 of [**Mac**]). The space ΔX is what one would call a simplex on X. To give an evidence we describe how one thinks of elements of ΔX as (free) convex combinations of points in X. For start, points in ΔK are convex combinations of elements of K whereas a point in $\mathsf{map}(K, X)$ identifies these elements with some points in X. Hence the points in the product $\mathsf{map}(K, X) \times \Delta K$ can be thought of as convex combinations of points in X together with a labeling of these points by elements of K. The coend then quotients out the relations we would expect: bijections identify all the possible labelings (so that we are left just with convex combinations of points in X), inclusions tell us that we can leave out any summand of the form $0 \cdot x$ and surjections give the relation $s \cdot x + t \cdot x = (s + t) \cdot x$. Note that for a finite discrete space K this agrees with our previous definition of ΔK.

REMARK. There is a different description of this construction. The functor $\Delta : \mathsf{Top} \to \mathsf{ConvTop}$ is the left adjoint of the forgetful functor $\mathsf{ConvTop} \to \mathsf{Top}$ defined on the category of convex topological spaces (a topological space X together with a map $I \times X \times X \to X$ thought of as the map $(t, x, y) \mapsto (1 - t)x + ty$ satisfying certain relations).

A fibrewise version of this construction for spaces $f : X \to Y$ over Y needs just a small modification: instead of allowing all convex combinations one only considers

combinations of points in a single fibre. In effect one replaces $\mathsf{map}(K, X)$ by X_Y^K, the subspace of those maps $g : K \to X$ for which $fg : K \to Y$ is constant

$$\Delta_Y X := \int^{K \in \mathcal{F}} X_Y^K \times \Delta K$$

We have the following pullback diagram (as $\Delta_Y X$ is a subspace of ΔX)

$$\begin{array}{ccc} \Delta_Y X & \longrightarrow & \Delta X \\ \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & \Delta Y \end{array}$$

$\mathcal{F}$ has a filtration by full subcategories $\mathcal{F}_n$ of sets of cardinality at most $n + 1$. Hence we get a filtration of $\Delta_Y X$ by the n-skeleta:

$$\Delta_Y X \cong \operatorname{colim}\left(X = \Delta_Y^0 X \hookrightarrow \cdots \hookrightarrow \Delta_Y^n X \hookrightarrow \cdots \right)$$

where

$$\Delta_Y^n X := \int^{K \in \mathcal{F}_n} X_Y^K \times \Delta K \cong \int^{K \in \mathcal{F}} X_Y^K \times \Delta^n K$$

All the maps $\Delta_Y^{n-1} X \hookrightarrow \Delta_Y^n X$ are easily seen to be closed inclusions[1] and

$$\Delta_Y^{(n)} X = \Delta_Y^n X - \Delta_Y^{n-1} X \cong X_Y^{(n+1)} \times_{\Sigma_{n+1}} \operatorname{int}\Delta^n \tag{2.1}$$

where $X_Y^{(n+1)}$ denotes the subspace of X_Y^{n+1} of $(n+1)$-tuples of distinct points in X (lying in one fibre over Y) and $\operatorname{int}\Delta^n$ denotes the interior of the standard n-dimensional simplex. From now on we assume that all the spaces admit a proper map $g : X \to \mathbb{R}_+$ (the nonnegative real numbers). Such spaces are necessarily locally compact and paracompact (and the converse is true for connected spaces).

LEMMA 2.1. *Let $K \subset \Delta X$ be a compact subset. Then $\operatorname{supp} K$, i.e. the closure of the set of points in X which appear in some element of K with a non-zero coefficient, is a compact subset.*

PROOF. For this proof we introduce a useful tool - an extension map

$$E : \mathsf{map}(X, \mathbb{R}) \to \mathsf{map}(\Delta X, \mathbb{R})$$

defined as an adjoint of

$$E^\sharp : \Delta X \times \mathsf{map}(X, \mathbb{R}) \to \mathbb{R}$$

sending $(\sum t_i x_i, g)$ to $\sum t_i g(x_i)$. It can be easily seen to be continuous.

Let us start with the proof now. Assume that $\operatorname{supp} K$ is not compact. Then there is a sequence of points $x_n \in X$ such that $g(x_n) \to \infty$ and such that x_n appears in some element of K with a nonzero coefficient t_n. Let $k : \mathbb{R} \to \mathbb{R}$ be any map such that $k(g(x_n)) > n/t_n$. Then $E(kg)$ is unbounded on K giving a contradiction. $\qquad\square$

---

[1] the idea is that one can think of $\Delta_Y^{n-1} X \hookrightarrow \Delta_Y^n X$ as $\int^{K \in \mathcal{F}_n} X_Y^K \times \Delta^{n-1} K \longrightarrow \int^{K \in \mathcal{F}_n} X_Y^K \times \Delta^n K$ and show explicitly that it is closed from the fact that the maps $f^* : X_Y^L \to X_Y^K$ are closed for $f : K \to L$ surjective.

COROLLARY 2.2. *Suppose that $f : X \to Y$ is a proper map such that the number of preimages $|f^{-1}(y)|$ is bounded on every compact subset of Y. Then the induced map $\Delta X \to \Delta Y$ is also proper.*

PROOF. Suppose that $K \subset \Delta Y$. Then $K \subset \Delta^n Y$ for some n. On a compact subset $\operatorname{supp} K \subset Y$, $|f^{-1}(y)| \leq k$ and finally $(\Delta f)^{-1}(K)$ is a closed subset of $\Delta^{kn}(f^{-1}(\operatorname{supp} K))$ which is a compact space. $\qquad \square$

Let us restrict ourselves to the case of a surjective[2] proper map $f : X \to Y$. As we noted earlier we have a pullback diagram

$$
\begin{array}{ccc}
Rf := \Delta_Y X & \lhook\joinrel\longrightarrow & \Delta X \\
\hat{f} \downarrow \quad & \lrcorner & \quad \downarrow \Delta f \\
Y & \lhook\joinrel\longrightarrow & \Delta Y
\end{array}
$$

and for simplicity we denote $Rf := \Delta_Y X$. It is easy to check that $\hat{f}$, as a pullback of a proper map, is again proper. Therefore Rf admits a proper map to $\mathbb{R}_+$. Moreover, the fibre of $\hat{f}$ at each point is a (finite-dimensional) simplex. In particular, by Vietoris-Begle Theorem C.4, $\hat{f}$ induces an isomorphism in Čech cohomology with compact supports. As we have seen above, Rf has a natural filtration by

$$
R^n f = \Delta_Y^n X = \int^{K \in \mathcal{F}} X_Y^K \times \Delta^n K
$$

We then get a spectral sequence with

$$
E_1^{pq} = \check{H}_c^{p+q}(R^p f, R^{p-1} f) \cong \check{H}_c^{p+q}(R^p f - R^{p-1} f)
$$

(the isomorphism coming from Proposition II.12.2. and II.10.2. of [**Bre**]). When $|f^{-1}(y)|$ is bounded on Y then the filtration is finite and the spectral sequence converges to $\check{H}_c^{p+q}(Rf) \cong \check{H}_c^{p+q}(Y)$. We have an identification (2.1)

$$
R^{(p)} f = R^p f - R^{p-1} f \cong X_Y^{(p+1)} \times_{\Sigma_{p+1}} \operatorname{int} \Delta^p
$$

and therefore, by the Thom isomorphism, $E_1^{pq} \cong \check{H}_c^q(X_Y^{[p+1]}; \operatorname{sign})$ where $X_Y^{[p+1]} = X_Y^{(p+1)}/\Sigma_{p+1}$ and sign denotes the system of coefficients on $X_Y^{[p+1]}$ given by the composition

$$
X_Y^{[p+1]} \longrightarrow B\Sigma_{p+1} \xrightarrow{B\operatorname{sign}} B\mathbb{Z}/2
$$

where the first map represents the principal Σ_{p+1}-bundle $X_Y^{(p+1)} \to X_Y^{[p+1]}$. Therefore we have a result:

THEOREM 2.3. *For a surjective finite-to-one proper map $f : X \to Y$ we have a spectral sequence*

$$
E_1^{pq} = \check{H}_c^q(X_Y^{[p+1]}; \operatorname{sign}) \Rightarrow \check{H}_c^{p+q}(Y)
$$

which converges if $|f^{-1}(y)|$ is bounded on Y. $\qquad \square$

---

[2]When applying the following to a non-surjective map we just need to replace Y by $\operatorname{im} f$ - as an image of a proper map it is a closed subspace and so it possesses a proper map to $\mathbb{R}_+$: simply restrict any proper map $Y \to \mathbb{R}_+$ to $\operatorname{im} f$.

Remark. There is a different way to define the topology on Rf (when $Rf \to Y$ is proper, the case that we considered). Remember our extension map from the proof of Lemma 2.1

$$E^\sharp : \Delta X \times \mathsf{map}(X, \mathbb{R}) \to \mathbb{R}$$

and consider its other adjoint

$$\widetilde{E} : \Delta X \to \mathsf{map}(\mathsf{map}(X, \mathbb{R}), \mathbb{R})$$

I do not know whether $\widetilde{E}$ expresses ΔX as a subspace of $\mathsf{map}(\mathsf{map}(X, \mathbb{R}), \mathbb{R})$ but I doubt it. Nevertheless for a locally compact Hausdorff (or completely regular for this matter) space X the restriction of $\widetilde{E}$ to X is a subspace inclusion

$$X \hookrightarrow \mathsf{map}(\mathsf{map}(X, \mathbb{R}), \mathbb{R})$$

Therefore we have a commutative diagram

$$
\begin{array}{ccccc}
Rf & \xrightarrow{\;g\;} & \widetilde{Rf} & \hookrightarrow & \mathsf{map}(\mathsf{map}(X, \mathbb{R}), \mathbb{R}) \\
\downarrow{\scriptstyle \hat{f}} & & \vdots{\scriptstyle \tilde{f}} & & \downarrow \\
Y & =\!=\!= & Y & \hookrightarrow & \mathsf{map}(\mathsf{map}(Y, \mathbb{R}), \mathbb{R})
\end{array}
$$

where $\widetilde{Rf}$ denotes the image of Rf in $\mathsf{map}(\mathsf{map}(X, \mathbb{R}), \mathbb{R})$, i.e. the same set Rf but with the subspace topology. As $\hat{f}$ is proper, so are $\tilde{f}$ and g. Then g is a bijective continuous proper map between locally compact Hausdorff spaces and is therefore a homeomorphism.

CHAPTER 3

A transversality theorem

In this chapter we present a transversality theorem for affine maps from topological affine spaces to sections of an affine bundle claiming that a certain subset is residual. For this to have some weight we prove that the affine spaces that we consider - the topologically finitely generated affine $\mathcal{R}$-modules - are Baire spaces, i.e. residual subsets are dense.

We will be considering maps $\varphi : P \to C^\infty(M, N)$. We denote by $\varphi^\sharp$ the adjoint

$$\varphi^\sharp : P \times M \to N$$

For the following let us denote by $C^\infty_\partial(M, N)$ the subspace of $C^\infty(M, N)$ of those maps f for which $f^{-1}(\partial N) = \partial M$.

LEMMA 3.1. *Let M, N be smooth manifolds and $A \subseteq N$ a submanifold without boundary such that either $A \subseteq N - \partial N$ or $A \subseteq \partial N$. Let there be given two open coverings: U of M and V of A. Let P be a topological space and $\varphi : P \to C^\infty_\partial(M, N)$ a continuous map where $C^\infty_\partial(M, N)$ is given the weak topology. Assume that for every $p_0 \in P$ and every $U \in$ U, $V \in$ V there is a finite dimensional manifold Q and a continuous map $g : Q \to P$ with p_0 in its image such that*

$$Q \times U \xrightarrow{g \times incl} P \times M \xrightarrow{\varphi^\sharp} N$$

is smooth and transverse to V. Then the set

$$\mathfrak{X} := \big\{ p \in P \mid \varphi(p) \pitchfork A \big\} \subseteq P$$

is residual in P.

PROOF. Following the proof of the Theorem 4.9. of Chapter 4 of [**GG**], let us cover M by a countable family of compact disks Y_i that have a neighbourhood $U_i \in$ U and at the same time we choose a covering of A by a countable family of compact subsets Z_j that have a neighbourhood $V_j \in$ V. Then the set $\mathfrak{X}$ is a countable intersection of the sets

$$\mathfrak{X}_{ij} := \big\{ p \in P \mid \varphi(p) \pitchfork Z_j \text{ on } Y_i \big\}$$

and it is enough to show that each $\mathfrak{X}_{ij}$ is open and dense. The set $\hat{\mathfrak{X}}_{ij}$ of maps $M \to N$ transverse to Z_j on Y_i is open in $C^\infty(M, N)$ and $\mathfrak{X}_{ij} = \varphi^{-1}(\hat{\mathfrak{X}}_{ij})$ so it is also open.

To prove the denseness we fix $p_0 \in P$ and choose a map $g : Q \to P$ with $p_0 = g(q_0)$ such that the map

$$h : Q \times U_i \to N$$

from the statement is smooth and transverse to V_j. By the parametric transversality theorem[1] the points $q \in Q$ for which $h(q,-) \pitchfork V_j$ is dense in Q. In particular q_0 lies in the closure of this set and hence p_0 lies in the closure of its image in P. But this image certainly lies in $\mathfrak{X}_{ij}$. $\qquad\square$

For the next corollary we denote by $\varphi^{(k)}$ the following map

$$\varphi^{(k)} : P \longrightarrow C^\infty(M,N) \longrightarrow C^\infty(M^{(k)}, N^k)$$

The second map sends f to $f^k|_{M^{(k)}}$ and it is clear that it is continuous in the weak topology.

COROLLARY 3.2. *Let M, N be smooth manifolds and $A \subseteq N^k$ a submanifold without boundary lying in a single depth $\partial_i N - \partial_{i+1} N$ of the boundary of N. Let there be given two open coverings: U of M and V of A. Let P be a topological space and $\varphi : P \to C^\infty_\partial(M,N)$ a continuous map where $C^\infty_\partial(M,N)$ is given the weak topology. Assume that for every $p_0 \in P$ and every $U \in \mathrm{U}$, $V \in \mathrm{V}$ there is a finite dimensional manifold Q and a continuous map $g : Q \to P$ with p_0 in its image such that*

$$Q \times U \xrightarrow{g \times incl} P \times M^{(k)} \xrightarrow{(\varphi^{(k)})^\sharp} N^k$$

is smooth and transverse to V. Then the set

$$\mathfrak{X} := \big\{ p \in P \mid \varphi^{(k)}(p) \pitchfork A \big\} \subseteq P$$

is residual in P. $\qquad\square$

PROPOSITION 3.3. *Let $\mathcal{V}$ be a topological affine space (over $\mathbb{R}$), M a smooth manifold and $E \to M$ a smooth finite dimensional affine bundle. Let $A \subseteq E$ be a submanifold. Let $\varphi : \mathcal{V} \to \Gamma E$ be a continuous affine map, where the set $\Gamma E \subseteq C^\infty(M,E)$ of sections of E is given the weak topology. We denote by φ_x the map*

$$\varphi^\sharp(-,x) : \mathcal{V} \to E_x$$

where E_x is the fibre of E over x. Let us assume that for each $x \in M$ the image $\varphi_x(\mathcal{V})$ is either the whole fibre E_x or is disjoint with A. Then the set

$$\mathfrak{X} := \big\{ v \in \mathcal{V} \mid \varphi(v) \pitchfork A \big\} \subseteq \mathcal{V}$$

is residual in $\mathcal{V}$.

PROOF. Because of the assumptions we can replace M by its open subset where φ_x is surjective. In effect we can therefore assume that $\varphi^\sharp$ is surjective.

As $\varphi_x : \mathcal{V} \to E_x$ is a surjective affine map and E_x is finite dimensional we can find a splitting. Let us denote the image of such a splitting by Q_0 and give it the Euclidean topology. Then the restricted map

$$Q_0 \times M \longrightarrow \mathcal{V} \times M \xrightarrow{\varphi^\sharp} E$$

is affine and moreover isomorphic on the fibre over x. Therefore there is a coordinate disk K around x such that this map is isomorphic over K. If $v \in V$, then taking the

---

[1]cf. Theorem 2.7. of Chapter 3 of [**Hir**]. For manifolds with boundary $h^{-1}(V_j)$ is still a submanifold.

affine hull $Q = \langle Q_0, v \rangle$ of $Q_0 \cup \{v\}$ (again with the Euclidean topology) we see that the hypothesis of the last lemma is satisfied. $\qquad\square$

The proper setup for the multi-version is the affine bundle

$$E_{\underline{M}}^{(k)} \longrightarrow M^{(k)}$$

which is just the restriction of $E^k \longrightarrow M^k$ to $M^{(k)}$. There is a canonical continuous affine map $\Gamma E \longrightarrow \Gamma\big(E_{\underline{M}}^{(k)}\big)$ and we denote the composition

$$\mathcal{V} \xrightarrow{\ \varphi\ } \Gamma E \longrightarrow \Gamma\big(E_{\underline{M}}^{(k)}\big)$$

by φ_k. As a corollary of the last proposition we get:

COROLLARY 3.4. *Let $\mathcal{V}$ be a topological affine space (over $\mathbb{R}$), M a smooth manifold and $E \to M$ a smooth finite dimensional affine bundle. Let $A \subseteq E_{\underline{M}}^{(k)}$ be a submanifold. Let $\varphi : \mathcal{V} \to \Gamma E$ be a continuous affine map, where $\Gamma E \subseteq \overline{C}^{\infty}(M, E)$ is given the weak topology. Let us assume that for each $(x_1, \ldots, x_k) \in M^{(k)}$ the image $(\varphi_{x_1} \times \cdots \times \varphi_{x_k})(\mathcal{V})$ is either the whole product $E_{x_1} \times \cdots \times E_{x_k}$ of fibres or is disjoint with A. Then the set*

$$\mathfrak{X} := \big\{v \in \mathcal{V} \ \big|\ \varphi_k(v) \pitchfork A\big\} \subseteq \mathcal{V}$$

is residual in $\mathcal{V}$. $\qquad\square$

PROPOSITION 3.5. *Under the assumptions of the last proposition let $v_i \in \mathrm{dir}\ \mathcal{V}$, $i = 1, \ldots, n$ be any elements of the underlying vector space of $\mathcal{V}$. Define a map*

$$\tilde{\varphi} : \mathcal{V} \to C^{\infty}(M \times \mathbb{R}^n, E)$$

or rather its adjoint

$$\tilde{\varphi}^{\sharp} : \mathcal{V} \times M \times \mathbb{R}^n \to E$$

by the formula

$$\tilde{\varphi}^{\sharp}(v, x, (\lambda_i)) = (\varphi(v + \lambda_1 v_1 + \cdots + \lambda_n v_n)(x))$$

Then the set

$$\mathfrak{X} := \big\{v \in \mathcal{V} \ \big|\ \tilde{\varphi}(v) \pitchfork A\big\} \subseteq \mathcal{V}$$

is residual in $\mathcal{V}$.

PROOF. Let $E \times \mathbb{R}^n \to M \times \mathbb{R}^n$ be the product bundle and define $\hat{\varphi} : \mathcal{V} \to \Gamma(E \times \mathbb{R}^n)$ by

$$\hat{\varphi}^{\sharp}(v, x, (\lambda_i)) = (\varphi(v + \lambda_1 v_1 + \cdots + \lambda_n v_n)(x), (\lambda_i))$$

Obviously $\mathfrak{X} = \big\{v \in \mathcal{V} \ \big|\ \hat{\varphi}(v) \pitchfork A \times \mathbb{R}^n\big\}$ and to apply the last proposition it is enough to verify that $\hat{\varphi}$ is continuous. We can express $\hat{\varphi}^{\sharp}$ as

$$\big(\big(\varphi(v) \times \mathrm{id}\big) + \big(\varphi(\lambda_1 v_1 + \cdots + \lambda_n v_n) \times \mathrm{id}\big)\big)(x, (\lambda_i))$$

and therefore $\hat{\varphi}$ is a sum of two terms the first of which is the composition

$$\mathcal{V} \xrightarrow{\ \varphi\ } \Gamma E \longrightarrow \Gamma(E \times \mathbb{R}^n)$$

while the second is constant. $\qquad\square$

Again we have a multi-version:

THEOREM 3.6. *Under the assumptions of the last corollary let $v_i \in \mathrm{dir}\ \mathcal{V}$, $i = 1, \ldots, n$ be any elements of the underlying vector space of $\mathcal{V}$. Define a map*

$$\tilde{\varphi}_k : \mathcal{V} \to C^\infty\left(M^{(k)} \times \mathbb{R}^n, E_{\underline{M}}^{(k)}\right)$$

or rather its adjoint

$$\tilde{\varphi}_k^\sharp : \mathcal{V} \times M^{(k)} \times \mathbb{R}^n \to E_{\underline{M}}^{(k)}$$

by the formula

$$\tilde{\varphi}_k^\sharp(v, (x_j), (\lambda_i)) = (\varphi(v + \lambda_1 v_1 + \cdots + \lambda_n v_n)(x_j))$$

Then the set

$$\mathfrak{X} := \left\{v \in \mathcal{V} \ \middle|\ \tilde{\varphi}_k(v) \pitchfork A\right\} \subseteq \mathcal{V}$$

is residual in $\mathcal{V}$.

PROPOSITION 3.7. *Let M be a smooth manifold. Then every topologically finitely generated affine $C^\infty(M, \mathbb{R})$-module is a Baire space, i.e. every residual subset is dense.*

PROOF. This is almost a triviality. Let $\mathcal{R} = C^\infty(M, \mathbb{R})$ and $\mathcal{V}$ be a topologically finitely generated affine $\mathcal{R}$-module. It is well-known that $\mathcal{R}E = C^\infty(M, E)$ is a Baire space. We have a quotient map

$$p : \mathcal{R}E \to \mathcal{V}$$

As it is essentially a quotient by the action of a group - the kernel of this map - it is an open map. Hence if $U_i \subseteq \mathcal{V}$, $i = 1, \ldots,$ are open dense subsets then so are $p^{-1}(U_i)$ and so

$$\mathcal{V} = p\left(\overline{\bigcap p^{-1}(U_i)}\right) \subseteq \overline{p\left(\bigcap p^{-1}(U_i)\right)} = \overline{\bigcap U_i}$$

$$\square$$

REMARK. This works well for both the strong and the weak topology on $\mathcal{R} = C^\infty(M, \mathbb{R})$ but note that in the strong topology while $\mathcal{R}$ is a topological ring it is not a topological $\mathbb{R}$-algebra unless M is compact.

A spectral sequence

This chapter is the heart of the whole thesis. It is based on the article [**Vas**] of Vassiliev. First we have to explain what our setup is. We have a (topologically finitely generated) affine $\mathcal{R}$-module $\mathcal{V}$ and its "representation" on an affine bundle E over M which is a certain map $\varphi : \mathcal{V} \to \Gamma E$ to the space of smooth sections of E.

Typically E is a jet prolongation of some affine bundle F and $\mathcal{V}$ consists either of all sections of E or of all integrable sections - those that come from sections of the bundle F (so that in this case one can think of $\mathcal{V}$ as ΓF). The next piece of data is a "prohibited" closed subset $A \subseteq E$. The main interest of the thesis is in the subspace $\mathcal{V}_A$ of $\mathcal{V}$ consisting of those $v \in \mathcal{V}$ for which $\varphi(v) \in \Gamma E$ is a section that does *not* intersect A.

We closely follow [**Vas**], [**Vas2**] and construct (in this abstract setting) a spectral sequence computing homology of $\mathcal{V}_A$. Here we have to assume that A is a stratified subset of E of codimension at least $\dim M + 1$. The main idea is to approximate $\mathcal{V}$ by finite dimensional affine subspaces D which "have good transversality and interpolation properties".

These properties ensure for example that for all $v \in D$ the section $\varphi(v)$ has only finitely many intersections with A. Let us denote by X the space of pairs (v, x) where $v \in D$ and $x \in M$ is a point such that the section $\varphi(v)$ meets A at x. Then we can reformulate this finiteness condition as the projection map $X \to D$ being finite-to-one. This is where the spectral sequence of Chapter 2 comes up. It converges to the cohomology of the image of this projection which is clearly $D - \mathcal{V}_A$. Alternatively, by Alexander duality, it converges to the (reduced) *homology* of $D \cap \mathcal{V}_A$. Depending on the "interpolation quality" of D one can identify a range of entries on the E_1-page.

With a bit of work one can glue these individual spectral sequences (for various finite dimensional affine subspaces D) to a single one where one is able to identify the whole E_1-page and it turns out that it does not depend on $\mathcal{V}$ too much. Therefore an affine $\mathcal{R}$-homomorphism $\alpha : \mathcal{U} \to \mathcal{V}$ is very likely to be a homology isomorphism provided that both spectral sequences (for $\mathcal{U}$ and $\mathcal{V}$) converge. Two criteria are given at the end of the chapter together with the above mentioned fundamental example of sections of a jet bundle and the submodule of integrable sections.

4.1. The spectral sequence

As was said in the introduction the main object of our study is a representation of a topological $\mathcal{R}$-module on a vector bundle $E \to M$ and the main example that we have in mind (and which should therefore satisfy our definition) is the map $j^r : \Gamma F \to$

$\Gamma(J^r F)$ sending a section of a vector bundle F to the corresponding integrable section of the jet prolongation $E = J^r F$.

Thinking of E as a vector bundle, ΓE becomes an $\mathcal{R}$-module *but* the map j^r is not $\mathcal{R}$-linear. An $\mathcal{R}$-homomorphism $\varphi : \mathcal{V} \to \Gamma E$ can be equivalently described as an $\mathbb{R}$-linear map that has the property $\varphi(I_x \mathcal{V}) \subseteq I_x \Gamma E$. The map j^r satisfies a weaker condition $j^r((I_x)^r \Gamma F) \subseteq I_x \Gamma E$. This is the most economical description of what our notion of a representation should be which is even sufficient for the construction of the spectral sequence. In next chapters however we will need more structure associated with representations. In order to describe it we make the following observations.

We consider the bundle $J^r(M, \mathbb{R}) \to M$ of algebras (over the trivial bundle $\mathbb{R} \times M \to M$) and denote $J^r \mathcal{R} = \Gamma(J^r(M, \mathbb{R}))$. The inclusion $\mathbb{R} \times M \subseteq J^r(M, \mathbb{R})$ then induces an algebra homomorphism $i^r : \mathcal{R} \to J^r \mathcal{R}$. Every vector bundle $E \to M$ has a canonical fibrewise action of $\mathbb{R} \times M \to M$ and this is how the action of $\mathcal{R}$ on ΓE arises. Suppose now that there is a fibrewise action of $J^r(M, \mathbb{R})$ on E extending the canonical action of $\mathbb{R} \times M$. Then ΓE becomes a $J^r \mathcal{R}$-module and restricting the action of $J^r \mathcal{R}$ along $i^r : \mathcal{R} \to J^r \mathcal{R}$ we recover the original $\mathcal{R}$-module structure of ΓE. On the other hand we also have an algebra homomorphism $j^r : \mathcal{R} \to J^r \mathcal{R}$ sending a section f to its r-jet prolongation $j^r f$ and we denote by $j^r \Gamma E$ the $\mathcal{R}$-module obtained by restriction along j^r. It is generally different from ΓE.

DEFINITION 4.1. By a j^r-*representation* of an $\mathcal{R}$-module $\mathcal{V}$ on a vector bundle E over M we mean an action of $J^r(M, \mathbb{R})$ on E as above together with an $\mathcal{R}$-linear map $\varphi : \mathcal{V} \to j^r \Gamma E$. Note that always $\varphi((I_x)^r \mathcal{V}) \subseteq I_x \Gamma E$ as the action is fibrewise.

By an *affine j^r-representation* of an affine $\mathcal{R}$-module $\mathcal{V}$ on an affine bundle E over M we mean an action of $J^r(M, \mathbb{R})$ on the underlying vector bundle of E together with an affine $\mathcal{R}$-homomorphism $\varphi : \mathcal{V} \to j^r \Gamma E$.

DEFINITION 4.2. For the purpose of this chapter we use the following definition of a stratified subset. Let M be a smooth manifold. We say that $X \subseteq M$ is a stratified subset if there is given a finite decomposition

$$X = \coprod_{i=0}^{k} X_i$$

of X into disjoint subsets X_i that are submanifolds of M with either $X_i \subseteq M - \partial M$ or $X_i \subseteq \partial M$ and such that each partial union $\bigcup_{i=0}^{j} X_i$ is closed in M, $j = 0, \dots, k$. We call this decomposition a stratification of X.

The advantage of this definition is that it is closed under transverse pullbacks, i.e. if $f : N \to M$ is a smooth map such that $f \pitchfork X_i$ for all $i = 0, \dots, k$ then $f^{-1}(X_i)$ provide a stratification of $f^{-1}(X)$.

On the other hand this definition is strong enough to say something about cohomological properties of X. We define the dimension $\dim X$ of X to be the maximal dimension of its stratum.

LEMMA 4.3. $\check{H}_c^n(X; \mathcal{A}) = 0$ *for all* $n > \dim X$ *and any coefficient system* $\mathcal{A}$ *on* X.

PROOF. We have a spectral sequence $E_2^{pq} = \check{H}_c^{p+q}(X_p; \mathcal{A}|_{X_p}) \Rightarrow \check{H}_c^{p+q}(X; \mathcal{A})$ associated with the filtration of X by $\bigcup_{i=0}^{j} X_i$ and by our assumptions this spectral sequence vanishes outside of $0 \le p + q \le \dim X$. As it converges the same holds for $\check{H}_c^{p+q}(X; \mathcal{A})$. $\square$

NOTATION. In this chapter let $\mathcal{V}$ a topologically finitely generated affine $\mathcal{R}$-module, $\xi : E \to M$ an affine bundle with e the dimension of the fibre, $\varphi : \mathcal{V} \to j^r \Gamma E$ an affine j^r-representation of $\mathcal{V}$ on E, $\hat{M}$ the subset of all points $x \in M$ for which φ_x is surjective, $A \subseteq E$ a closed stratified subset such that outside of $\hat{M}$ the image of $\varphi_x : \mathcal{V} \to E_x$ is disjoint with A. We denote

$$c = \big(\text{codimension of } A \text{ in } E\big) - m$$

From now on we will assume that $c > 0$ or in other words that the codimension of A in E is at least $m + 1$. We are interested in the space

$$\mathcal{V}_A := \{v \in \mathcal{V} \mid \text{im}(\varphi(v)) \cap A = \emptyset\}$$

here particularly in its homology groups.

CONSTRUCTION 4.4. We recall the multi-version of φ

$$\varphi_k : \mathcal{V} \xrightarrow{\varphi} \Gamma E \longrightarrow \Gamma\big(E_{\underline{M}}^{(k)}\big)$$

We will also use its adjoint $\varphi_k^\sharp : \mathcal{V} \times M^{(k)} \to E_{\underline{M}}^{(k)}$. We claim that for $(x_1, \dots, x_k) \in \hat{M}^{(k)}$ the map

$$(\varphi_k)_{(x_1,\dots,x_k)} : \mathcal{V} \longrightarrow \big(E_{\underline{M}}^{(k)}\big)_{(x_1,\dots,x_k)} \cong E_{x_1} \times \cdots \times E_{x_k}$$

is surjective. This is because for any $i \in \{1, \dots, k\}$ and for any $w \in E_{x_i}$ there is $v \in \mathcal{V}$ with $\varphi(v)(x_i) = w$; multiplying v by any function λ which is 1 near x_i and 0 near the remaining points we get

$$\varphi_k(\lambda v)(x_1, \dots, x_k) = (0, \dots, 0, \overset{\overset{i}{\downarrow}}{w}, 0, \dots, 0)$$

We also have a multi-version $A_{\underline{M}}^{(k)}$ of the stratified subset A in $E_{\underline{M}}^{(k)}$, its codimension is $k(c + m)$ and on the fibres over points outside of $\hat{M}^{(k)}$ the image of $\varphi_k^\sharp$ is disjoint with $A_{\underline{M}}^{(k)}$.

PROPOSITION 4.5. *There is a sequence of finite dimensional affine subspaces $D_d \subseteq \mathcal{V}$ satisfying*

(1$_d$) *For each k the map $\varphi_k^\sharp : D_d \times M^{(k)} \to E_{\underline{M}}^{(k)}$ is transverse to (each stratum of) A_k.*
(2$_d$) *$D_d \pitchfork S_{dr}(\mathcal{V})$.*
(3) *$D_d \subseteq D_{d+1}$ and the union $\bigcup_d D_d$ is dense in $\mathcal{V}$.*

PROOF. First note that conditions (2$_d$) and (3) depend only on the direction spaces of D_d. Theorem 1.15 ensures, for each d, an existence of a finite dimensional

D'_d satisfying (2_d). It is also well-known[1] that a sequence of finite dimensional D''_d satisfying (3) exists. Therefore

$$D'''_d = D'_1 + \cdots + D'_d + D''_d$$

is clearly a sequence of direction spaces satisfying both (2_d) and (3). According to Theorem 3.6 we can find $v \in \mathcal{V}$ such that $D_d = v + D'''_d$ will fulfill even (1_d): the number of conditions imposed on such a v is countable (one for each d and k) and each such condition is satisfied by a residual subset.   □

Let us try to explain now what these conditions are good for. We start with the condition (3) which roughly says that D_d approximate $\mathcal{V}$ well. This is made more precise in the following lemma.

LEMMA 4.6. *Whenever D_d is an increasing sequence of affine subspaces of a locally convex topological affine space $\mathcal{V}$ such that $D_\infty := \bigcup_d D_d$ is dense in $\mathcal{V}$ then for any open subset $U \subseteq \mathcal{V}$ we have the following isomorphisms*

$$\mathrm{colim}\ H_*(D_d \cap U) \cong H_*(D_\infty \cap U) \cong H_*(U)$$

PROOF. The idea of the proof is as follows. Define $H_*^{\mathrm{aff}}(X)$ for any open subspace X of a locally convex topological vector space in the same way as singular homology but only allowing affine maps $\Delta^n \to X$. We prove that the natural inclusion induces an isomorphism $H_*^{\mathrm{aff}}(X) \cong H_*(X)$. To do so, for a covering U of X by open convex subsets, we introduce $H_*^{\mathrm{aff}}(X, \mathrm{U})$ where image of each simplex has to lie in one of the elements of U. It is a standard fact that the inclusion induces an isomorphism $H_*(X, \mathrm{U}) \cong H_*(X)$ and the same holds for the affine version hence we are left to show that in the diagram

$$
\begin{array}{ccc}
H_*^{\mathrm{aff}}(X, \mathrm{U}) & \xrightarrow{\ \cong\ } & H_*^{\mathrm{aff}}(X) \\
\downarrow & & \downarrow \\
H_*(X, \mathrm{U}) & \xrightarrow{\ \cong\ } & H_*(X)
\end{array}
$$

the map $H_*^{\mathrm{aff}}(X, \mathrm{U}) \to H_*(X, \mathrm{U})$ is an isomorphism. One can easily check that the inverse is induced by "straightening" the simplices, i.e. by replacing every singular simplex by the unique affine simplex having the same vertices. Hence we can replace the singular homology H_* in the statement by its affine version H_*^{aff}. This proves the first isomorphism. The second one could be proved by an easy observation that one can perturb slightly (without changing the homology class) any affine chain in U to one in $D_\infty \cap U$.   □

This leads us to investigating $H_*(D_d \cap \mathcal{V}_A)$. The conditions (1_d) and (2_d) that we impose on these spaces will allow us to construct a spectral sequence for it. Hence let us fix a finite dimensional affine subspace $D \subseteq \mathcal{V}$ satisfying both (1_d) and (2_d). The proof of the last lemma also shows that $H_*(D \cap \mathcal{V}_A)$, being isomorphic to a purely algebraic group $H_*^{\mathrm{aff}}(D \cap \mathcal{V}_A)$, does not depend on the topology of D as long as this

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[1]It is certainly well-known for free $\mathcal{R}$-modules $\mathcal{R}E = C^\infty(M, E)$ of finite rank and this property clearly passes to quotients.

topology makes D into a locally convex topological vector space in which $\mathcal{V}_A$ is open. Therefore we can and will assume that the topology on D is the Euclidean topology.

Let us denote by $X \subset D \times M$ the stratified subset of points (v, x) for which $\varphi_x(v) \in A$. We denote by π the composition $X \hookrightarrow D \times M \to D$. Now we can draw some consequences of (1_d). First we need a multi-version of the stratified subset X. For every k we have the following diagram where the square is a pullback square

$$
\begin{array}{ccc}
 & X_D^{(k)} \longrightarrow & A_{\underline{M}}^{(k)} \\
\pi_k \swarrow \quad \big\uparrow \quad \lrcorner & & \big\uparrow \\
D \longleftarrow D \times M^{(k)} \xrightarrow{\varphi_k^\sharp} & & E_{\underline{M}}^{(k)}
\end{array}
\tag{4.1}
$$

(for $k = 1$ we are getting $X_D^{(k)} = X$ and $\pi_k = \pi$). The condition (1_d) then implies that $X_D^{(k)}$ is a Σ_k-equivariantly stratified subset of dimension $\dim X_D^{(k)} = \dim D - ck$. This will be important later, for now we only need to know that for $k > \dim D$, $X_D^{(k)} = \emptyset$ or in other words $|\pi^{-1}(v)| \le \dim D$ for every $v \in D$. Theorem 2.3 provides a spectral sequence

$$E_1^{pq}(D) = \check{H}_c^{p+q}(R^{(p)}\pi) \cong \check{H}_c^q(X_D^{[p+1]}; \mathrm{sign})$$

that converges to

$$\check{H}_c^{p+q}(\mathrm{im}(\pi)) \cong \widetilde{H}_{\dim D - (p+q) - 1}(D \cap \mathcal{V}_A)$$

(by Alexander duality). We will now identify $\check{H}_c^q(X_D^{[p+1]}; \mathrm{sign})$ using the condition (2_d). Assuming that $p < d$ the map $\varphi_{p+1}^\sharp : D \times \hat{M}^{(p+1)} \longrightarrow E_{\hat{M}}^{(p+1)}$ is fibrewise surjective. This is because the projection on the fibre over $(x_0, \ldots, x_p)$ has kernel

$$\varphi^{-1}\big(I_{x_0} \cdots I_{x_p}(\Gamma E)\big) \supseteq (I_{x_0})^r \cdots (I_{x_p})^r \mathcal{V}$$

and $D \pitchfork (I_{x_0})^r \cdots (I_{x_p})^r \mathcal{V} \in S_{(p+1)r}(\mathcal{V})$. Note that then

$$\varphi_{p+1}^\sharp : D \times \hat{M}^{(p+1)} \longrightarrow E_{\hat{M}}^{(p+1)}$$

is a fibrewise surjective affine map between affine bundles over $\hat{M}^{(p+1)}$ and so it is itself an affine bundle. By restricting this bundle to A_{p+1} we obtain a Σ_{p+1}-equivariant affine bundle $X_D^{(p+1)} \to A_{\hat{M}}^{(p+1)}$ as in the diagram (4.1) and by further taking Σ_{p+1}-orbits an affine bundle

$$h_{p+1} : X_D^{[p+1]} \to A_{\hat{M}}^{[p+1]}$$

By the Thom isomorphism

$$\check{H}_c^q(X_D^{[p+1]}; \mathrm{sign}) \cong \check{H}_c^{q-t}\left(A_{\hat{M}}^{[p+1]}; w_1(h_{p+1}) + \mathrm{sign}\right)$$

where $t = \dim D - e(p+1)$. As we mentioned earlier, this is under the assumptions that $p < d$.

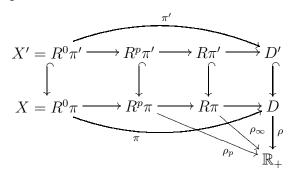NOTE. One can express $w_1(h_{p+1})$ in terms of $w_1(\xi)$ as follows. We have the following composition

$$H_*(\hat{M}^{[p+1]}) \xrightarrow{\mathrm{tr}} H_*(\hat{M}^{(p+1)})^{\Sigma_{p+1}} \xrightarrow{\mathrm{pr}} H_*(\hat{M})$$

where the first map is a transfer map for the covering $\hat{M}^{(p+1)} \to \hat{M}^{[p+1]}$ and the second map is a projection on a factor (does not matter which one). Applying $\mathrm{Hom}(-, \mathbb{Z}/2)$ we get a map in cohomology

$$H^1(\hat{M}; \mathbb{Z}/2) \longrightarrow H^1(\hat{M}^{[p+1]}; \mathbb{Z}/2) \longrightarrow H^1(A_{\underline{\hat{M}}}^{[p+1]}; \mathbb{Z}/2)$$

the second map being induced by the obvious map $A_{\underline{\hat{M}}}^{[p+1]} \to \hat{M}^{[p+1]}$. The image of $w_1(\xi)$ is $w_1(h_{p+1}) + e \cdot \mathrm{sign}$.

Our aim now is to glue the spectral sequences for all D_d into one and discuss its convergence. First we deal with some naturality issues for a single spectral sequence as above. Suppose that $D' \subseteq D$ and that D' also satisfies (1_d) and (2_d). We have the following diagram of spaces

Here $\rho : D \to \mathbb{R}_+$ is any seminorm on D with kernel D' (equivalently a norm on D/D'). Thinking of the middle row as a diagram of spaces over $\mathbb{R}_+$ via ρ the top row consists of fibres (over 0) of these spaces.

PROPOSITION 4.7. *There is a natural way of defining $i_p^!$ and $i_{(p)}^!$ in such a way that $i_\infty^! = i^!$ and the following diagram commutes (a morphism of exact couples extending $i^!$ on the colimit)*

$$
\begin{array}{ccccccc}
\check{H}_c^{*-1}(R^{p-1}\pi') & \longrightarrow & \check{H}_c^*(R^{(p)}\pi') & \longrightarrow & \check{H}_c^*(R^p\pi') & \longrightarrow & \check{H}_c^*(R^{p-1}\pi') \\
\downarrow{\scriptstyle i_{p-1}^!} & & \downarrow{\scriptstyle i_{(p)}^!} & & \downarrow{\scriptstyle i_p^!} & & \downarrow{\scriptstyle i_{p-1}^!} \\
\check{H}_c^{*+\delta-1}(R^{p-1}\pi) & \longrightarrow & \check{H}_c^{*+\delta}(R^{(p)}\pi) & \longrightarrow & \check{H}_c^{*+\delta}(R^p\pi) & \longrightarrow & \check{H}_c^{*+\delta}(R^{p-1}\pi)
\end{array}
$$

where $\delta = \dim D - \dim D'$. Moreover, if $p < d$ then $i_{(p)}^!$ is the Thom isomorphism.

PROOF. The idea is as follows. The map $i^! : \check{H}_c^*(R\pi') \to \check{H}_c^{*+\delta}(R\pi)$ can be obtained by taking a colimit $\varepsilon \to 0$ of

$$\check{H}_c^*(R\pi') \leftarrow \check{H}_c^*(\rho_\infty^{-1}[0,\varepsilon]) \to \check{H}_c^{*+\delta}(\rho_\infty^{-1}([0,\varepsilon],\{\varepsilon\})) \cong$$
$$\cong \check{H}_c^{*+\delta}(\rho_\infty^{-1}(\mathbb{R}_+,[\varepsilon,\infty))) \to \check{H}_c^{*+\delta}(R\pi)$$

which makes the first arrow an isomorphism. This is proved in the Appendix D in Theorem D.3. The second map is a cup product with the pullback along $R\pi \to D$ of the generator of $\check{H}^\delta(\rho^{-1}([0,\varepsilon],\{\varepsilon\})$. We can now define $i_p^!$ and $i_{(p)}^!$ by the same formula replacing ∞ by p or (p) and the commutativity of the diagram then follows from various naturality properties of the maps involved.

Taking a different seminorm one can construct a comparison map from the above diagram for ρ and ε to the same diagram for ρ' and ε'. It is easy to see that in the colimit this map is an isomorphism - its inverse is constructed in exactly the same way. Therefore our construction does not depend on the seminorm ρ.

The last part follows from the fact that in this case $R^{(p)}\pi' \hookrightarrow R^{(p)}\pi$ is an inclusion of (the image of) a section of an affine bundle and in this case the above constructed map is precisely the Thom isomorphism. $\qquad\square$

This ensures that we get a morphism of the corresponding spectral sequences. First we have to rewrite the spectral sequence in a homological way by transformation $p \leftrightarrow -p-1$ and $q \leftrightarrow \dim D - q$

$$E^1_{pq}(D) = \check{H}^{\dim D - (p+q) - 1}_c(R^{(-p-1)}\pi) \cong \check{H}^{\dim D - q}_c(X^{[-p]}_D; \mathrm{sign})$$

converging to $\widetilde{H}_{p+q}(D \cap \mathcal{V}_A)$. Stably (when $p > -d-1$) we have

$$E^1_{pq}(D) \cong \check{H}^{-ep-q}_c(A^{[-p]}_{\underline{\hat{M}}}; w_1(h_{-p}) + \mathrm{sign})$$

Now we can define a colimit spectral sequence

$$E^1_{pq}(\mathcal{V}) = \operatorname*{colim}_d \check{H}^{\dim D_d - (p+q) - 1}_c(R^{(-p-1)}\pi_d) \Rightarrow \widetilde{H}_{p+q}(\mathcal{V}_A) \qquad (4.2)$$

using Proposition 4.7. Also from this proposition it is clear that

$$E^1_{pq}(\mathcal{V}) \cong \check{H}^{-ep-q}_c(A^{[-p]}_{\underline{\hat{M}}}; w_1(h_{-p}) + \mathrm{sign})$$

for all p and q. As we discussed before $E^1_{pq}(D)$ always converges. Now we want to give some sufficient conditions for the convergence of $E^1_{pq}(\mathcal{V}) \Rightarrow \widetilde{H}_{p+q}(\mathcal{V}_A)$.

PROPOSITION 4.8. *If $E^1_{pq}(\mathcal{V}) = 0$ then the spectral sequence converges. In other words in this case $\widetilde{H}_*(\mathcal{V}_A) = 0$.*

PROOF. Up to a change in the indices $\widetilde{H}_*(D_d \cap \mathcal{V}_A)$ is identified with $\check{H}^*_c(R\pi_d)$ and for $d' \gg d$ the latter is mapped to the stable part of $\check{H}^*_c(R\pi_{d'})$ which by our assumptions is 0. In other words the map $\widetilde{H}_*(D_d \cap \mathcal{V}_A) \to \widetilde{H}_*(D_{d'} \cap \mathcal{V}_A)$ is 0 for $d' \gg d$. As $\widetilde{H}_*(\mathcal{V}_A) \cong \operatorname{colim} \widetilde{H}_*(D_d \cap \mathcal{V}_A)$ this clearly proves the claim. $\qquad\square$

PROPOSITION 4.9. *If $c > 1$, or in other words if the codimension of A in E is at least $m+2$, then the spectral sequence $E^1_{pq}(\mathcal{V}) \Rightarrow \widetilde{H}_{p+q}(\mathcal{V})$ converges.*

PROOF. The proof is similar. Clearly $E^1_{pq}(D_d) \cong \check{H}^{\dim D_d - q}_c(X^{[-p]}_{D_d}; \mathrm{sign})$ is 0 for $p > 0$. Hence we assume $p \leq 0$ and we recall that $X^{(-p)}_{D_d}$ is a Σ_{-p}-equivariantly stratified subset of $D_d \times M^{(-p)}$ of dimension at most $\dim D_d + cp$. Therefore we also get $E^1_{pq}(D_d) = 0$ for $q < -cp$. Consequently for a fixed n and for $d \gg 0$, $r \gg 0$ both $\widetilde{H}_n(D_d)$ and $\operatorname{Tot}\big(E^r_{**}(D_d)\big)_n = \bigoplus_{p+q=n} E^r_{pq}(D_d)$ do not depend on d and r. One concludes that the colimit spectral sequence converges from the convergence of each $E^1_{pq}(D_d)$. $\qquad\square$

THEOREM 4.10. *Suppose that $c > 1$ and that $\alpha : \mathcal{U} \subseteq \mathcal{V}$ is a topological submodule such that $(\varphi\alpha)_x$ is still surjective for all $x \in \hat{M}$. Then the map $\alpha_A : \mathcal{U}_A \to \mathcal{V}_A$ is a homology isomorphism.*

PROOF. We have a map of spectral sequences[2] $E^r_{pq}(\mathcal{U}) \to E^r_{pq}(\mathcal{V})$ and on the E^1-page this map is an isomorphism. Hence so is the map on the limits. $\square$

4.2. Examples

The following is a fundamental example.

EXAMPLE 4.11. Let $F \to M$ be a finite dimensional affine bundle and set $E = J^r F$. Then there is an obvious action of $J^r(M, \mathbb{R})$ on E and we have a topological $\mathcal{R}$-module $\mathcal{V} = j^r \Gamma E$ (which is topologically finitely generated by Example 1.44). Obviously $\mathcal{U} = \Gamma F$ can be identified via j^r with an $\mathcal{R}$-submodule of $\mathcal{V}$ of integrable sections. Clearly $\varphi = \mathrm{id} : \mathcal{V} \to \Gamma E$ is surjective on fibres as is φj^r. Therefore by the last theorem we know that if $c > 1$ then the inclusion map $(\Gamma F)_A \hookrightarrow (\Gamma(J^r F))_A$ is a homology isomorphism (this will be improved to having acyclic fibres in Theorem 6.1).

EXAMPLE 4.12. To illustrate what the spaces appearing in the spectral sequence $E^1_{pq} = E^1_{pq}(\mathcal{V})$ are let us take $\mathcal{V} = \Gamma E$, the space of sections of a vector bundle $\xi : E \to M$ of dimension at least $\dim M + 2$ and let $A = M$ be the zero section. Then with respect to the representation $\mathrm{id} : \mathcal{V} \to \Gamma E$ we have $\hat{M} = M$ and

$$E^1_{pq} \cong \check{H}^{-ep-q}_c(A^{[-p]}_{\hat{M}}; w_1(h_{-p}) + \mathrm{sign}) \cong H^{-ep-q}_c(M^{[-p]}; (pr \cdot \mathrm{tr})^* w_1(\xi) + (e+1)\mathrm{sign})$$

converging to the reduced homology of the space of sections of the associated unit sphere bundle.

---

[2]We are cheating here a bit. But it is not difficult to arrange the subspaces D_d in $\mathcal{U}$ and $\mathcal{V}$ in such a way that α preserves them. Also later we prove independently a stronger result.

CHAPTER 5

Constructing the homotopy fibre

In the previous chapter we saw that an affine $\mathcal{R}$-homomorphism $\alpha : \mathcal{U} \to \mathcal{V}$ induces a homology isomorphism $\alpha_A : \mathcal{U}_A \to \mathcal{V}_A$ if it is "enough surjective". In this chapter we make the first step towards the proof of a stronger result that all the homotopy fibres of α_A are acyclic (have zero homology). This step consists of constructing the homotopy fibres in our realm of j^r-representations of topologically finitely generated affine modules. A simple computation in the next chapter then shows that the first page of the basic spectral sequence (4.2) is zero.

In the first section we construct the underlying module and show that it is topologically finitely generated. In the next section the j^r-representation is described while the biggest part is devoted to the proof that what we get is indeed the homotopy fibre.

The actual construction is best understood on the example of the space $\mathcal{V} = \Gamma(F)$ of sections of an affine bundle F and its submodule $\mathcal{U}$. Classically the homotopy fibre would be a certain subspace of the space $\mathsf{map}(I, \mathcal{V})$ of continuous paths in $\mathcal{V}$. Remembering that $\mathcal{V} = \Gamma(F)$ we can identify the path space $\mathsf{map}(I, \mathcal{V})$ with some subspace of the space $\Gamma^0(F \times I)$ of continuous sections of $F \times I \to M \times I$. We follow the same idea but take only smooth sections. This corresponds in abstract to an extension of scalars. We have a topological algebra $\mathcal{R}^I = C^\infty(M \times I, \mathbb{R})$ and an inclusion map $\mathcal{R} \hookrightarrow \mathcal{R}^I$. Our space of sections of $F \times I$ is then nothing but $\mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{V}$.

The topological tensor product is not as well-behaved as is the algebraic one in general. Its basic properties on the category of locally convex topological vector spaces are gathered in Appendix A. In Appendix B a proof that $\mathcal{R}$ is locally convex can be found. As locally convex topological vector spaces are closed under products and quotients every topologically finitely generated $\mathcal{R}$-module is also locally convex. Therefore we do not lose anything in assuming that all $\mathcal{R}$-modules in this chapter are locally convex.

5.1. The algebraic part

We consider the smooth manifold $M \times I$ and set $\mathcal{R}^I = C^\infty(M \times I, \mathbb{R})$. There is a natural inclusion $\varepsilon : \mathcal{R} \subseteq \mathcal{R}^I$ induced by the projection $M \times I \to M$. For any $t \in I$ one gets a splitting $res_t : \mathcal{R}^I \to \mathcal{R}$ of the above inclusion $\mathcal{R} \subseteq \mathcal{R}^I$ induced by

$$M \cong M \times \{t\} \subseteq M \times I$$

Therefore every topological $\mathcal{R}^I$-module is canonically a topological $\mathcal{R}$-module and for every topological $\mathcal{R}$-module $\mathcal{V}$ we have an induced topological $\mathcal{R}^I$-module $\mathcal{V}^I := \mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{V}$ together with a split inclusion $\varepsilon : \mathcal{V} \hookrightarrow \mathcal{V}^I$. Also if $\mathcal{V}$ is topologically finitely

generated then so is $\mathcal{V}^I$. Indeed, tensoring the quotient map $\mathcal{R}E \to \mathcal{V}$ with $\mathcal{R}^I$ yields a quotient map

$$\mathcal{R}^I E \cong \mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{R}E \to \mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{V} = \mathcal{V}^I$$

LEMMA 5.1. *If we denote the I-coordinate function by t then there is a split short exact sequence*
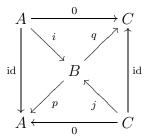
$$0 \longrightarrow \mathcal{R}^I \xrightarrow{t(t-1)\cdot} \mathcal{R}^I \xrightarrow{(res_0, res_1)^T} \mathcal{R} \times \mathcal{R} \longrightarrow 0$$

of $\mathcal{R}$-modules.

NOTE. As the category of topological $\mathcal{R}$-modules (or topological vector spaces) is not an abelian category we have to be a little careful here:

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$$

is a short exact sequence if A is the kernel (in the sense of topological $\mathcal{R}$-modules) of q and C the cokernel of i. In other words if it is a short exact sequence of $\mathcal{R}$-modules such that A has the subspace topology and C the quotient topology induced from B. The formulation of a split short exact sequence we are going to use here is that of a commutative diagram

such that $ip + jq = \text{id}$. In such a situation we say that B is a biproduct of A and C.

PROOF. The second map has a section sending (f_0, f_1) to $f(x,t) = (1-t)f_0(x) + tf_1(x)$. Now we construct a section of the first map. For $(x,t) \in M \times I$ we define a 2-simplex $\sigma_{(x,t)} : \Delta^2 \to M \times I$ to be

$$\sigma_{(x,t)} = [(x,0), (x,t), (x,1)]$$

and then for $f \in \mathcal{R}^I$ we set

$$\mathcal{J}(f,x,t) := \frac{1}{2} \cdot \int_{\Delta^2} f''_{tt} \sigma_{(x,t)}$$

The map $p : \mathcal{R}^I \to \mathcal{R}^I$ defined by $p(f)(x,t) = \mathcal{J}(f,x,t)$ is continuous and by Lemma 1.19 has the property that

$$t(t-1)p(f)(x,t) = f(x,t) - (1-t)f(x,0) - tf(x,1)$$

Therefore it is a splitting of the "multiplication by $t(t-1)$" map (as this multiplication map is clearly injective). The same formula proves that these splittings describe $\mathcal{R}^I$ as a biproduct of $\mathcal{R}^I$ and $\mathcal{R} \times \mathcal{R}$. $\qquad\square$

COROLLARY 5.2. *Let $\mathcal{V}, \mathcal{V}_0, \mathcal{V}_1$ be topologically finitely generated $\mathcal{R}$-modules, let $\alpha_i : \mathcal{V}_i \to \mathcal{V}$ be $\mathcal{R}$-homomorphisms. Then the $\mathcal{R}^I$-module $\mathcal{W}$ defined as the pullback*

$$
\begin{array}{ccc}
\mathcal{W} & \longrightarrow & \mathcal{V}_0 \times \mathcal{V}_1 \\
\downarrow & & \downarrow{\scriptstyle \alpha_0 \times \alpha_1} \\
\mathcal{V}^I & \xrightarrow{(res_0, res_1)^T} & \mathcal{V} \times \mathcal{V}
\end{array}
$$

is also topologically finitely generated. Here the $\mathcal{R}^I$-module structures on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V}_0 \times \mathcal{V}_1$ are induced by $\mathcal{R}^I \xrightarrow{(res_0, res_1)} \mathcal{R} \times \mathcal{R}$.

PROOF. Starting with the split short exact sequence from the previous lemma we tensor it with $\mathcal{V}$ to get

$$
0 \longrightarrow \mathcal{V}^I \xrightarrow{t(t-1)\cdot} \mathcal{V}^I \xrightarrow{(res_0, res_1)} \mathcal{V} \times \mathcal{V} \longrightarrow 0
$$

Taking the pullback along $\alpha_0 \times \alpha_1$ we get a split short exact sequence of $\mathcal{R}$-modules

$$
0 \longrightarrow \mathcal{V}^I \xrightarrow{i} \mathcal{W} \longrightarrow \mathcal{V}_0 \times \mathcal{V}_1 \longrightarrow 0
$$

Hence if $j : \mathcal{V}_0 \times \mathcal{V}_1 \to \mathcal{W}$ is the splitting and $\varphi : \mathcal{R}E \to \mathcal{V}$ and $\varphi_i : \mathcal{R}E_i \to \mathcal{V}_i$ are topological presentations then the following is a topological presentation

$$
(i\varphi^I, \overline{j\varphi_0}, \overline{j\varphi_1}) : \mathcal{R}^I E \times \mathcal{R}^I E_0 \times \mathcal{R}^I E_1 \longrightarrow \mathcal{W}
$$

of $\mathcal{W}$. Here $\overline{j\varphi_i}$ is the unique $\mathcal{R}^I$-linear extension of the $\mathcal{R}$-homomorphsisms $j\varphi_i : \mathcal{R}E_i \to \mathcal{W}$ to $(\mathcal{R}E_i)^I \cong \mathcal{R}^I E_i$. $\square$

NOTE. The same is true for affine $\mathcal{R}$-modules and affine maps between them. The pullback is clearly nonempty and so we can reduce to the linear case by choosing a common origin (take the images of any element in the pullback). As the forgetful functor $U : \mathcal{R}$-mod $\to$ Aff-$\mathcal{R}$-mod preserves limits[1] the affine case then follows from the linear version.

5.2. The topological part

Suppose that $\varphi : \mathcal{V} \to \Gamma E$ is a j^r-representation. We construct a j^r-representation of $\mathcal{V}^I$ on $E \times I$ as follows. There is a canonical homomorphism

$$
J^r(M \times I, \mathbb{R}) \to J^r(M, \mathbb{R}) \times I
$$

of bundles (of algebras) over $M \times I$ which simply forgets some of the jet information. On the other hand we have a product action

$$
(J^r(M, \mathbb{R}) \times I) \underset{M \times I}{\times} (E \times I) \to E \times I
$$

Composing these two maps gives the desired action of $J^r(M \times I, \mathbb{R})$ on $E \times I$. There is an obvious map $\varepsilon : \Gamma E \to \Gamma(E \times I)$ (sending a section $f : M \to E$ to the section $f \times \mathrm{id} : M \times I \to E \times I$ constant in the I direction). This map is $\mathcal{R}$-linear with respect to the j^r-structures. Therefore there is a unique $\mathcal{R}^I$-linear map $(j^r \Gamma E)^I \to j^r \Gamma(E \times I)$

---

[1]There is an adjunction $F :$ Aff-$\mathcal{R}$-mod $\rightleftarrows \mathcal{R}$-mod $: U$ where F sends an affine $\mathcal{R}$-module $\mathcal{V}$ to an affine $\mathcal{R}$-module spanned freely by $\mathcal{V}$ and an extra point 0 serving as an origin (this is a general fact about comma categories - the forgetful functor $c \downarrow \mathcal{C} \to \mathcal{C}$ has a left adjoint $d \mapsto (c \hookrightarrow d \coprod c)$).

giving a factorization $j^r \Gamma E \to (j^r \Gamma E)^I \to j^r \Gamma(E \times I)$ of ε and we define φ^I to be the composition $\mathcal{V}^I \to (j^r \Gamma E)^I \to \Gamma(E \times I)$.

$$\begin{CD} \mathcal{V} @>\varepsilon>> \mathcal{V}^I @>=>> \mathcal{V}^I \\ @V\varphi VV @VVV @VV\varphi^I V \\ \Gamma E @>>> (j^r \Gamma E)^I @>>> \Gamma(E \times I) \end{CD}$$

Now we need few auxiliary constructions. Proving that they are all given by continuous maps would destroy the flow of this section and so these proofs are only given in Appendix B.

Let $\lambda : (I, \partial I) \to (I, \partial I)$ be a smooth function. We define a reparametrization map

$$\lambda^* : \mathcal{R}^I \to \mathcal{R}^I$$

by $f(x, t) \mapsto f(x, \lambda(t))$. It is easily seen to be a continuous $\mathcal{R}$-homomorphism. Moreover thinking of $\mathcal{R}^I$ as a bundle over $\mathcal{R} \times \mathcal{R}$ via (res_0, res_1) it is a fibrewise map (not necessarily over id but there is only a discrete choice of base maps). If λ_0, λ_1 are two such maps with $\lambda_0 = \lambda_1$ on ∂I then the maps λ_0^* and λ_1^* are homotopic through reparametrization maps and in particular by a fibrewise homotopy. For an $\mathcal{R}$-module $\mathcal{V}$ we define a reparametrization map

$$\lambda^* : \mathcal{V}^I = \mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{V} \xrightarrow{\lambda^* \otimes \mathrm{id}} \mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{V} = \mathcal{V}^I$$

and observe that for any j^r-representation of $\mathcal{V}$ the diagram

$$\begin{CD} \mathcal{V}^I @>>> \Gamma(E \times I) \\ @V\lambda^* VV @VV\lambda^* V \\ \mathcal{V}^I @>>> \Gamma(E \times I) \end{CD}$$

commutes. Let us denote by $(\mathcal{R}^I)^{\hat{n}}$ the following subset of $(\mathcal{R}^I)^n$

$$(\mathcal{R}^I)^{\hat{n}} = \left\{ (f_1, \ldots, f_n) \in (\mathcal{R}^I)^n \mid res_1 f_i = res_0 f_{i+1}, i = 1, \ldots, n-1 \right\}$$

i.e. the space of n-tuples of functions that "can be concatenated". The problem is of course that one cannot concatenate them straight away, the result would not be smooth. We have a projection

$$(\sigma, \tau) : (\mathcal{R}^I)^{\hat{n}} \to \mathcal{R} \times \mathcal{R}$$

sending $(f_1, \ldots, f_n)$ to $(res_0 f_1, res_1 f_n)$ (here σ stands for the source map and τ for the target map). We think of $(\mathcal{R}^I)^{\hat{n}}$ as a space over $\mathcal{R} \times \mathcal{R}$ via this map. Let $0 = s_0 < \cdots < s_n = 1$ be a sequence of numbers and $\lambda_1, \ldots, \lambda_n : I \to I$ smooth nondecreasing maps with $\lambda_i = 0$ near 0 and $\lambda_i = 1$ near 1 (but where we only require $\lambda_1(0) = 0$ and $\lambda_n(1) = 1$ to allow $\mathrm{id} = \mu_1((0, 1), \mathrm{id})$). We define a concatenation map

$$\mu_n = \mu_n((s_i), (\lambda_i)) : (\mathcal{R}^I)^{\hat{n}} \longrightarrow \mathcal{R}^I$$

by sending (f_i) to the concatenation of the maps $\lambda_i^* f_i$ shrunk to $M \times [s_{i-1}, s_i]$. It is clearly an $\mathcal{R}$-linear map over $\mathcal{R} \times \mathcal{R}$. We also have the canonical inclusion $\varepsilon : \mathcal{R} \to \mathcal{R}^I$ and a reverse map

$$\iota : \mathcal{R}^I \xrightarrow{(1-t)^*} \mathcal{R}^I$$

(which is not a map over $\mathcal{R} \times \mathcal{R}$). All the maps $\mu_n((s_i), (\lambda_i))$ (for all possible s_i, λ_i but with n fixed) are homotopic through maps of the form μ_n (and in particular over $\mathcal{R} \times \mathcal{R}$). One easily verifies that $\mu_n(\mu_{k_1} \times \cdots \times \mu_{k_n})$ is of the form $\mu_{k_1 + \cdots + k_n}$; $\mu_{n+1}(\varepsilon\sigma, \mathrm{id})$ and $\mu_{n+1}(\mathrm{id}, \varepsilon\tau)$ are of the form μ_n. In particular $\mu_2(\mathrm{id} \times \mu_2) \simeq \mu_2(\mu_2 \times \mathrm{id})$ through maps of the form μ_3 and similarly $\mu_2(\varepsilon\sigma, \mathrm{id}) \simeq \mathrm{id}$, $\mu_2(\mathrm{id}, \varepsilon\tau) \simeq \mathrm{id}$. Also

$$\mu_2(f, \iota f) = (\mu_2(t, 1-t))^* f$$

and thus $\mu_2(\mathrm{id}, \iota) \simeq \sigma$ over $\mathcal{R} \times \mathcal{R}$ and through reparametrization maps, similarly $\mu_2(\iota, \mathrm{id}) \simeq \tau$.

Now we extend these maps to $\mathcal{V}^I$ by taking the tensor product with $\mathcal{V}$. We identify $(\mathcal{R}^I)^{\hat{n}} \otimes_{\mathcal{R}} \mathcal{V}$ with the obvious generalization $(\mathcal{V}^I)^{\hat{n}}$ of $(\mathcal{R}^I)^{\hat{n}}$ using Lemma 5.1. We get maps $\mu_n : (\mathcal{V}^I)^{\hat{n}} \longrightarrow \mathcal{V}^I$, $\varepsilon : \mathcal{V} \to \mathcal{V}^I$ and $\iota : \mathcal{V}^I \to \mathcal{V}^I$. They commute with the representation $\varphi^I : \mathcal{V}^I \to \Gamma(E \times I)$. The crucial property of these maps on $\Gamma(E \times I)$ is that for any subset $A \subseteq E$ these maps preserve

$$\Gamma(E \times I)_{A \times I} = \{f \in \Gamma(E \times I) \mid \mathrm{im}\, f \cap (A \times I) = \emptyset\}$$

where for ε this is understood as $\varepsilon(\Gamma E_A) \subseteq \Gamma(E \times I)_{A \times I}$. Therefore the maps on $\mathcal{V}^I$ preserve

$$(\mathcal{V}^I)_{A \times I} = (\varphi^I)^{-1} \Gamma(E \times I)_{A \times I}$$

THEOREM 5.3. *The following map is a Serre fibration if A is closed*

$$p : (\mathcal{V}^I)_{A \times I} \xrightarrow{(res_0, res_1)} \mathcal{V}_A \times \mathcal{V}_A$$

PROOF. The map

$$\mathcal{V}^I \xrightarrow{(res_0, res_1)} \mathcal{V} \times \mathcal{V}$$

is a projection on a direct summand by Lemma 5.1. As both $(\mathcal{V}^I)_{A \times I}$ and $\mathcal{V}_A \times \mathcal{V}_A$ are open subsets (when A is closed) we see that for every $u_0 \in \mathcal{V}_A$ there is a neighbourhood U of u_0 and a section

$$j_{u_0} : U \cong \{u_0\} \times U \to (\mathcal{V}^I)_{A \times I}$$

with $j_{u_0}(u_0) = \varepsilon(u)$. Therefore if $(u_0, v_0) \in \mathcal{V}_A \times \mathcal{V}_A$ and U, V are the neighbourhoods as above we can define two maps over $U \times V$

$$
\begin{aligned}
p^{-1}(U \times V) &\to U \times V \times p^{-1}(u_0, v_0) \\
U \times V \times p^{-1}(u_0, v_0) &\to p^{-1}(U \times V)
\end{aligned}
$$

The first map sends w to

$$(res_0 w, \ res_1 w, \ j_{u_0}(res_0 w) * w * \iota(j_{v_0}(res_1 w)))$$

and the second sends (u, v, w) to $\iota(j_{u_0}(u)) * w * j_{v_0}(v)$ where we use the notation with $*$ to denote the multiplication μ_n.

They are easily seen to be homotopy inverse to each other over $U \times V$. Therefore p has the weak homotopy lifting property. As an open subset of a fibration it is also a microfibration. These two properties together easily imply that it is a Serre fibration[2].   □

---

[2]For a map $f : D^k \times I \to \mathcal{V}_A \times \mathcal{V}_A$ and a partial lift L_0 over $D^k \times 0$ there is a fibrewise homotopy h from L_0 to a lift that can be extended to L_1 defined on $D^k \times I$. We think of h and L_1 as giving a partial lift of $D^k \times I \times I \to D^k \times I \xrightarrow{f} \mathcal{V}_a \times \mathcal{V}_A$ over $D^k \times 0 \times I \cup D^k \times I \times 1$. By a microfibration

NOTE. We have an affine version of all the constructions. To define

$$\mathcal{V}^I = \mathcal{R}^I \otimes_{\mathcal{R}} \mathcal{V}$$

we just choose an origin in $\mathcal{V}$, take the ordinary tensor product and then forget the origin of the result. The canonical inclusion $\varepsilon : \mathcal{V} \to \mathcal{V}^I$ as well as the maps μ_n, ι do not depend on this choice. The same is true for the induced j^r-representation

$$\varphi^I : \mathcal{V}^I \to \Gamma(E \times I)$$

of an affine j^r-representation $\varphi : \mathcal{V} \to \Gamma E$: the choice of an origin in $\mathcal{V}$ gives a section in E which then makes E into a vector bundle. After extending to $M \times I$ we forget the origin/section and the result is independent of our choice. Clearly Theorem 5.3 remains true in this case.

Let $\alpha : \mathcal{U} \to \mathcal{V}$ be an affine $\mathcal{R}$-homomorphism, $\varphi : \mathcal{V} \to \Gamma E$ an affine j^r-representation, $A \subseteq E$ a closed subset. By $\alpha_A : \mathcal{U}_A \to \mathcal{V}_A$ we denote the restriction of α to the open subset $\mathcal{U}_A$. As this is defined via the representation $\varphi\alpha : \mathcal{U} \to \Gamma E$ its image necessarily lies in $\mathcal{V}_A$.

THEOREM 5.4. *In the notation from above let $v \in \mathcal{V}_A$ be a point and let $\mathcal{W}$ be the pullback*

$$
\begin{array}{ccc}
\mathcal{W} & \longrightarrow & \{v\} \times \mathcal{U} \\
\downarrow & & \downarrow {\scriptstyle incl \times \alpha} \\
\mathcal{V}^I & \xrightarrow{(res_0, res_1)^T} & \mathcal{V} \times \mathcal{V}
\end{array}
$$

Then the homotopy fibre $\mathrm{hofib}_v\alpha_A$ *of* $\alpha_A : \mathcal{U}_A \to \mathcal{V}_A$ *over v has the weak homotopy type of $\mathcal{W}_{A \times I}$ defined in terms of the affine j^r-representation $\psi : \mathcal{W} \to \mathcal{V}^I \xrightarrow{\varphi^I} \Gamma(E \times I)$.*

PROOF. We have a diagram where all the squares are pullback squares

$$
\begin{array}{ccccc}
 & & \mathcal{W} \longrightarrow \{v\} \times \mathcal{U}_A & \longrightarrow & \{v\} \\
 & & \downarrow \quad\lrcorner \qquad \downarrow & & \downarrow \\
\mathcal{U}_A & \xrightarrow{\simeq} \mathcal{P} \xrightarrow{F} \mathcal{V}_A \times \mathcal{U}_A & \xrightarrow{projection} & \mathcal{V}_A \\
 & G\downarrow \quad\lrcorner \qquad \downarrow {\scriptstyle id \times \alpha_A} & & \\
 & (\mathcal{V}^I)_{A \times I} \xrightarrow{(res_0, res_1)^T} \mathcal{V}_A \times \mathcal{V}_A & &
\end{array}
$$

The maps denoted by $\twoheadrightarrow$ are fibrations and the map denoted by $\simeq$ is a homotopy equivalence constructed from the universal property of a pullback from maps $\mathcal{U}_A \xrightarrow{(\alpha_A, id)} \mathcal{V}_A \times \mathcal{U}_A$ and $\mathcal{U}_A \xrightarrow{\varepsilon\alpha_A} (\mathcal{V}^I)_{A \times I}$ (there is a deformation retraction

$$(\lambda_s^* G, (res_s G, pr_{\mathcal{U}_A} F)) : \mathcal{P} \longrightarrow \mathcal{P} = ((\mathcal{V}^I)_{A \times I}) \times_{(\mathcal{V}_A \times \mathcal{V}_A)} (\mathcal{V}_A \times \mathcal{U}_A)$$

whose main part is $\lambda_s^* : (\mathcal{V}^I)_{A \times I} \to (\mathcal{V}^I)_{A \times I}$, $s \in I$, with $\lambda_s(t) = t + (1 - t)s$. As the composition across the middle row is α_A we see that $\mathcal{W}$ is the homotopy fibre $\mathrm{hofib}_v\alpha_A$. $\qquad \square$

property there is a neighbourhood $D^k \times [0, \varepsilon] \times I \cup D^k \times I \times [1 - \varepsilon, 1]$ and a lift L_2 over it extending both h and L_1. We define $L : D^k \times I \to (\mathcal{V}^I)_{A \times I}$ to be $L_2(x, t, t/\varepsilon)$ for $t \leq \varepsilon$ and $L_2(x, t, 1)$ for $t \geq \varepsilon$.

LEMMA 5.5. *For the affine j^r-representation ψ from the last theorem*

$$\psi_{(x,t)}(\mathcal{W}) = \left\{ \begin{array}{ccc} \varphi_x(v) & for & t = 0 \\ \varphi_x(\mathcal{V}) & for & 0 < t < 1 \\ (\varphi\alpha)_x(\mathcal{U}) & for & t = 1 \end{array} \right.$$

PROOF. We have a diagram

$$
\begin{array}{ccc}
\mathcal{W} & \longrightarrow & \{v\} \times \mathcal{U} \\
\downarrow & & \downarrow {\scriptstyle incl \times \alpha} \\
\mathcal{V} \xleftarrow{res_t} \mathcal{V}^I & \xrightarrow{(res_0, res_1)^T} & \mathcal{V} \times \mathcal{V} \\
\downarrow {\scriptstyle \varphi} \quad \downarrow {\scriptstyle \varphi^I} & & \downarrow {\scriptstyle \varphi \times \varphi} \\
\Gamma E \xleftarrow{res_t} \Gamma(E \times I) & \xrightarrow{(res_0, res_1)^T} & \Gamma E \times \Gamma E
\end{array}
$$

This proves easily the claims for $t = 0$ and $t = 1$. Also it is clear that for $0 < t < 1$ the image can be at most $\varphi_x(\mathcal{V}) = (\varphi^I)_{(x,t)}(\mathcal{V}^I)$. But according to the proof of Corollary 5.2 we have a decomposition $\overline{\mathcal{W}} \cong \overline{\mathcal{V}^I} \times \overline{\{v\}} \times \overline{\mathcal{U}}$ of the underlying $\mathcal{R}$-modules and the linear part of the composition $\mathcal{V} \subseteq \mathcal{V}^I \xrightarrow{\psi} \Gamma(E \times I)$ maps v to a section

$$(x, t) \mapsto t(t-1)\varphi_x(v)$$

and as $t(t-1) \neq 0$ the image is the same as that of φ_x.   □

CHAPTER 6

The main theorem

This short chapter is devoted to outlining the relation between cohomology of a configuration space of a product of two spaces and cohomology of the two individual configuration spaces. The answer is not at all satisfactory but it suffices for proving (under some assumptions) the acyclicity of the homotopy fibre of $\alpha_A : \mathcal{U}_A \longrightarrow \mathcal{V}_A$ for an affine $\mathcal{R}$-homomorphism $\alpha : \mathcal{U} \longrightarrow \mathcal{V}$. An example is given at the end.

Let X be a locally compact Hausdorff space. By $X^{[d]}$ we denote the configuration space of d distinct unordered points in X, i.e. the quotient $X^{(d)}/\Sigma_d$ where $X^{(d)} \subseteq X^d$ is the subspace of injective maps $d \to X$. In this chapter we will be interested in the Čech cohomology with compact supports of the space $(X \times Z)^{[d]}$ where Z is another locally compact Hausdorff space. Note that there is a canonical map

$$f : (X \times Z)^{[d]} \to \mathrm{S}_d(X) = X^d/\Sigma_d$$

We want to apply Leray spectral sequence to this map to get some cohomological information about the configuration space $(X \times Z)^{[d]}$ and so we need to identify the fibres of f. Let $Y = \{(x_1, k_1), \dots, (x_n, k_n)\} \in \mathrm{S}_d(X)$. The fibre $f^{-1}(Y)$ is then obviously homeomorphic to $Z^{[k_1]} \times \cdots \times Z^{[k_n]}$. Let us suppose now that for all $k > 0$ and all locally constant sheaves $\mathcal{A}$ on X

$$\check{H}_c^*(Z^{[k]}; \mathcal{A}) = 0$$

as is for example the case for $Z = \mathbb{R}_+$, the closed half-line. Here $(\mathbb{R}_+)^{[k]}$ is topologically a k-simplex with all its $(k-1)$-faces but one removed. Hence its one-point compactification is homeomorphic to the k-dimensional disk D^k and thus contractible. The Künneth formula then says that for all $k_1, \dots, k_n > 0$ and all locally constant sheaves $\mathcal{A}$ we have

$$\check{H}_c^*(Z^{[k_1]} \times \cdots \times Z^{[k_n]}; \mathcal{A}) = 0$$

Therefore in the Leary spectral sequence of f with compact supports we have $E_2^{pq} = 0$ and therefore also $\check{H}_c^*((X \times Z)^{[d]}; \mathcal{A}) = 0$ for any locally compact Hausdorff space X and any locally constant sheaf $\mathcal{A}$ on $(X \times Z)^{[d]}$.

Now we apply these ideas to the homotopy fibre of α_A for some affine $\mathcal{R}$-homomorphism $\alpha : \mathcal{U} \to \mathcal{V}$, affine j^r-representation $\varphi : \mathcal{V} \to \Gamma E$ and a stratified subset $A \subseteq E$ of codimension at least $\dim M + 2$. Recall that by Theorem 5.4 this homotopy fibre can be identified with $\mathrm{hofib}_v \alpha_A \simeq \mathcal{W}_{A \times I}$. See this theorem for the explanation of $\mathcal{W}$ and the affine j^r-representation ψ via which $\mathcal{W}_{A \times I}$ is defined.

Provided that $(\varphi\alpha)_x$ is surjective on $\hat{M}$, Lemma 5.5 guarantees that $\psi_{(x,t)}$ is either surjective or disjoint with $A \times I$ and that $\widehat{M \times I} = \hat{M} \times (0, 1]$. As the codimension of $A \times I$ in $E \times I$ is at least $\dim(M \times I) + 1$ we have a spectral sequence (4.2) for the

affine j^r-representation ψ from the theorem

$$E^1_{pq}(\mathcal{W}) \cong \check{H}^{-ep-q}_c \left((A \times I)^{[-p]}_{\underline{M \times I}}; w_1(h_{-p}) + \mathrm{sign} \right) \Rightarrow \widetilde{H}_{p+q}(\mathcal{W}_{A \times I})$$

For the case of the configuration space of $\hat{M} \times (0,1]$ situation is the same as above: the fibres of $(A \times I)^{[-p]}_{\hat{M} \times (0,1]} \to A^{-p}_{\underline{\hat{M}}}/\Sigma_{-p}$ are products of $(0,1]^{[k]}$ and we are getting $E^1_{pq}(\mathcal{W}) = 0$. According to Proposition 4.8 this spectral sequence converges and thus $\widetilde{H}_*(\mathrm{hofib}_v \alpha_A) = 0$. Therefore we have a theorem

Theorem 6.1. *Let M be a compact smooth manifold, let $\alpha : \mathcal{U} \to \mathcal{V}$ be an affine $\mathcal{R}$-homomorphism between two topologically finitely generated affine $\mathcal{R}$-modules, let $\varphi : \mathcal{V} \to \Gamma E$ be an affine j^r-representation and $A \subseteq E$ a stratified subset of codimension at least $\dim M + 2$ such that outside the set*

$$\hat{M} = \{x \in M \mid (\varphi \alpha)_x \text{ is surjective}\}$$

we have $\mathrm{im}\, \varphi_x \cap A = \emptyset$. Then each homotopy fibre $\mathrm{hofib}_v \alpha_A$ of α_A is acyclic, i.e. $\widetilde{H}_*(\mathrm{hofib}_v \alpha_A) = 0$. $\qquad\square$

We finish with an example (and its refinement) of use of our main Theorem 6.1.

Example 6.2. Let us consider smooth functions $M \to \mathbb{R}$ and let $A \subseteq J^3(N, \mathbb{R})$ be the complement of the set of 3-jets which have at most A_2-singularity (i.e. are either regular, nondegenerate critical or have a singularity of type A_2). In other words the complement of A consists precisely of those jets which take in some coordinate chart the form

$$f(x_1, \ldots, x_m) = c \pm x_1^k \pm x_2^2 \pm \cdots \pm x_m^2$$

with $1 \leq k \leq 3$. Then according to Igusa's theorem (Theorem 9.1 of [**Igu**]) the canonical map

$$(j^3)_A : C^\infty(M, \mathbb{R})_A \to \Gamma(J^3(M, \mathbb{R}))_A$$

is $(\dim M)$-connected and therefore its homotopy fibres are $(\dim M - 1)$-connected. In particular when $\dim M > 1$ they are simply connected and by Theorem 6.1 and Whitehead's theorem they are weakly contractible making $(j^3)_A$ a weak homotopy equivalence.

Example 6.3. To demonstrate an example in which the setting of topological $\mathcal{R}$-modules can be easily applied we modify the previous example slightly by considering a closed submanifold $N \subseteq M$ which is either neat or $N = \partial M$ and fix a Morse function $g : M \to \mathbb{R}$ (or just its germ at N). Next we consider the following topologically finitely generated affine $\mathcal{R}$-modules

$$\begin{aligned} \mathcal{U} &= \{f \in C^\infty(M, \mathbb{R}) \mid j^3_z f = j^3_z g \ \forall z \in N\} \\ \mathcal{V} &= \{s \in \Gamma(J^3(M, \mathbb{R})) \mid s(z) = j^3_z g \ \forall z \in N\} \end{aligned}$$

According to Theorem 6.1 the homotopy fibres of the restricted jet prolongation $(j^3)_A : \mathcal{U}_A \to \mathcal{V}_A$ (with A from the previous example) are again acyclic. Here $\mathcal{U}_A$ is the space of functions $M \to \mathbb{R}$ with at most A_2-singularities which agree with g up to order 3 at N and similarly for $\mathcal{V}_A$ and we obtain a relative version of the previous example.

APPENDIX A

Locally convex topological vector spaces

This appendix serves as a source of results about locally convex topological vector spaces for the purpose of the thesis. The main source used was [**Trè**] where one can find everything (in a great more detail) with the exception of Theorem A.2.

We work here with vector spaces over the field $\mathbb{R}$ of real numbers. A topological vector space $\mathcal{V}$ is called *locally convex* if every neighbourhood of 0 contains a convex neighbourhood of 0.

A subset $V \subseteq \mathcal{V}$ is called *balanced* if $rV \subseteq V$ for all $|r| \leq 1$ and *absorbing* if for every $v \in \mathcal{V}$ there is $r > 0$, such that the line segment from $-v$ to v is contained in rV. It is called a *barrel* if it closed, convex, balanced and absorbing. Equivalently, $\mathcal{V}$ is locally convex if every neighbourhood V of 0 contains a barrel neighbourhood of 0.

A seminorm on a vector space $\mathcal{V}$ is a function $\rho : \mathcal{V} \to \mathbb{R}$ such that

  (a) $\rho(v) \geq 0$
  (b) $\rho(v + v') \leq \rho(v) + \rho(v')$
  (c) $\rho(\lambda \cdot v) = |\lambda| \cdot \rho(v)$

Every continuous seminorm on a topological vector space $\mathcal{V}$ gives a closed ball $B_\rho = \rho^{-1}[0, 1]$. They are barrel neighbourhoods of 0. If on the other hand V is a barrel neighbourhood of 0 in $\mathcal{V}$ then there exists a continuous seminorm $\rho : \mathcal{V} \to \mathbb{R}$, the so called Minkowski functional, such that $V = \rho^{-1}[0, 1]$. It is defined by the formula

$$\rho(v) = \inf\{\lambda \in \mathbb{R}_+ \mid 1/\lambda \cdot v \in V\}$$

Therefore, if $\mathcal{V}$ is locally convex its topology can be completely described by continuous seminorms.

A family $\mathcal{P}$ of continuous seminorms on a LCTVS $\mathcal{V}$ is called a *basis of continuous seminorms* if for any continuous seminorm σ on $\mathcal{V}$ there is $r > 0$ and $\rho \in \mathcal{P}$ such that $\sigma \leq r\rho$, or equivalently if the closed ε-balls $B_\rho(\varepsilon) = \rho^{-1}[0, \varepsilon]$ form a neighbourhood basis of 0. In such a case we say that the family $\mathcal{P}$ *define* the topology of $\mathcal{V}$.

For example, if $\mathcal{V}$ is a (not necessarily locally convex) TVS, the family of all continuous seminorms define some locally convex topology on $\mathcal{V}$. It is always weaker than the one that we started with and this procedure provides a left adjoint of the forgetful functor LCTVS $\hookrightarrow$ TVS (hence LCTVS is a reflective subcategory of TVS).

If $\mathcal{U}$, $\mathcal{V}$ are two vector spaces, $\alpha : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \otimes \mathcal{V}$ the canonical map and $U \subseteq \mathcal{U}$, $V \subseteq \mathcal{V}$ any convex subsets we define their convex tensor product $U \otimes V \subseteq \mathcal{U} \otimes \mathcal{V}$ to be the convex hull of the set $\alpha(U \times V) = \{u \otimes v \mid u \in U, v \in V\}$. If ρ, σ are seminorms on $\mathcal{U}$, $\mathcal{V}$ we define a seminorm $\rho \otimes \sigma : \mathcal{U} \otimes \mathcal{V} \to \mathbb{R}$ by the requirement

$B_{\rho \otimes \sigma} = B_\rho \otimes B_\sigma$. Therefore if $\mathcal{U}$ and $\mathcal{V}$ are both LCTVS we can define a locally convex topology on their tensor product $\mathcal{U} \otimes \mathcal{V}$ via the seminorms $\rho \otimes \sigma$ where ρ, σ range over all continuous seminorms on $\mathcal{U}$, $\mathcal{V}$ (or for that purpose ranging over any bases of seminorms on the respective spaces). Together with this topology we call $\mathcal{U} \otimes \mathcal{V}$ the projective tensor product of $\mathcal{U}$ and $\mathcal{V}$.

PROPOSITION A.1. *If $\mathcal{U}$ and $\mathcal{V}$ are both LCTVS then the projective topology on $\mathcal{U} \otimes \mathcal{V}$ is the strongest LCTVS topology for which $\alpha : \mathcal{U} \times \mathcal{V} \to \mathcal{U} \otimes \mathcal{V}$ is continuous. It enjoys the following universal property: every continuous bilinear map $\beta : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ to a LCTVS $\mathcal{W}$ factorizes uniquely through α as*

$$\beta : \mathcal{U} \times \mathcal{V} \xrightarrow{\alpha} \mathcal{U} \otimes \mathcal{V} \xrightarrow{\gamma} \mathcal{W}$$

with γ a continuous linear map. In other words there is a natural isomorphism

$$\mathsf{LCTVS}(\mathcal{U} \otimes \mathcal{V}, \mathcal{W}) \cong \mathsf{Bilin}(\mathcal{U}, \mathcal{V}; \mathcal{W})$$

PROOF. The first part follows straight from the definition. According to the algebraic properties of a tensor product we only need to show that the continuity of β implies the continuity of γ. Therefore let $W \subseteq \mathcal{W}$ be any convex neighbourhood of 0. By continuity of β there are ρ, σ such that $\beta(B_\rho \times B_\sigma) \subseteq W$. As W is convex it also contains

$$\mathrm{ch}(\beta(B_\rho \times B_\sigma)) = \mathrm{ch}(\gamma\alpha(B_\rho \times B_\sigma)) = \gamma(\mathrm{ch}\,\alpha(B_\rho \times B_\sigma)) = \gamma(B_{\rho \otimes \sigma})$$

where ch denotes the convex hull.   □

THEOREM A.2. *The category LCTVS of locally convex topological vector spaces has the following structure*

(i) LCTVS *together with $\otimes$ is a symmetric monoidal category.*
(ii) *The tensor product functor $\mathcal{U} \otimes -$ commutes with finite colimits.*
(iii) *If $\mathcal{U}, \mathcal{V}, \mathcal{W} \in$ LCTVS, P is a locally compact space and $P \times (\mathcal{U} \times \mathcal{V}) \to \mathcal{W}$ is a continuous pointwise $\mathbb{R}$-bilinear map then the induced map $P \times (\mathcal{U} \otimes \mathcal{V}) \to \mathcal{W}$ is also continuous. In particular the tensor product preserves homotopies.*

PROOF. The associativity and commutativity of $\otimes$ follow from the description of the topology on the tensor product, e.g. the topology on $(\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W} \cong \mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W})$ is generated by neighbourhoods of 0 of the form $(B_\rho \otimes B_\sigma) \otimes B_\tau = B_\rho \otimes (B_\sigma \otimes B_\tau)$.

Tensoring a biproduct diagram with $\mathcal{U}$ clearly produces again a biproduct diagram (mere additivity of the tensor product suffices) so $\mathcal{U} \otimes -$ preserves finite coproducts. The fact that $\mathcal{U} \otimes -$ preserves cokernels follows from the corresponding property of the product $\mathcal{U} \times -$.

To prove (iii) we use adjunction to reduce everything to showing that $\mathsf{map}(P, \mathcal{W})$ is locally convex. But a subbasis for the topology of $\mathsf{map}(P, \mathcal{U})$ is given by convex sets $\{f : P \to \mathcal{U} \mid f(K) \subseteq U\}$ with K ranging over all compact and U over all convex open subsets.   □

NOTE. Any locally convex topological $\mathbb{R}$-algebra $\mathcal{A}$ is a monoid in the monoidal structure on LCTVS and topological $\mathcal{A}$-modules are precisely $\mathcal{A}$-modules in the monoidal category sense. Consequently we can use monoidal category techniques to deal with

$\mathcal{A}$-modules. For simplicity we assume that $\mathcal{A}$ is commutative. For $\mathcal{A}$-modules $\mathcal{U}$ and $\mathcal{V}$ we get a tensor product $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{V}$ as a coequalizer

$$\mathcal{U} \otimes \mathcal{A} \otimes \mathcal{V} \rightrightarrows \mathcal{U} \otimes \mathcal{V}$$

the two maps being the two structure maps. In this way $\mathcal{A}$-modules become a symmetric monoidal category (here the right exactness of the tensor product is necessary in order to define the action of $\mathcal{A}$ on $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{V}$).

Suppose that $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous algebra homomorphism between commutative topological $\mathbb{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$ (i.e. a homomorphism of commutative monoids in the monoidal category LCTVS). Then the forgetful functor $\varphi^* : \mathcal{B}\text{-mod} \to \mathcal{A}\text{-mod}$ has a left adjoint $\mathcal{B} \otimes_{\mathcal{A}} -$.

APPENDIX B

Differential topology, function spaces

The purpose of this appendix is to give a brief overview of the differential topology used and to prove some auxiliary results that would, if included in the main text, disturb the flow of exposition. The two main sources used were [**GG**] and [**Hir**].

Let $E \to M$ be a smooth fibre bundle. There are two kinds of topologies on the space $\Gamma(E)$ of smooth sections, the weak (or sometimes called compact-open) topology and strong (or Whitney) topology defined for each degree $0 \leq r \leq \infty$ of differentiability. We start with the definition for $r = 0$. The weak topology on the space $\Gamma^0(E)$ of continuous sections of E is just the usual compact-open topology. We denote the resulting space by $\Gamma^0_W(E)$. The basis for the strong topology on $\Gamma^0(E)$ is indexed by open subsets U of E for which we have a generating open subset

$$\{s \in \Gamma^0(E) \mid \operatorname{im}(s) \in U\} \subseteq \Gamma^0(E)$$

The resulting topological space is denoted by $\Gamma^0_S(E)$. For finite r and $* = W, S$ we obtain the topology on $\Gamma^r_*(E)$ via the jet prolongation map

$$j^r : \Gamma^r(E) \longrightarrow \Gamma^0_*(J^r E) \tag{B.1}$$

and the above definition for $r = 0$. More precisely this map is injective and we give $\Gamma^r_*(E)$ the subspace topology. In the diagram

$$
\begin{array}{ccc}
\Gamma^{r+1}_*(E) & \lhook\joinrel\longrightarrow & \Gamma^0_*(J^{r+1}E) \\
\Big\downarrow & & \Big\downarrow \\
\Gamma^r_*(E) & \lhook\joinrel\longrightarrow & \Gamma^0_*(J^r E)
\end{array}
$$

the dashed arrow is continuous by the properties of subspaces (or one can say that this arrow exists in Top) and is clearly the canonical inclusion. We define the topology on $\Gamma^\infty_*(E)$ as the limit topology

$$\Gamma^\infty_*(E) \cong \lim_{r \to \infty} \Gamma^r_*(E)$$

In other words one just takes all the open subsets from all $\Gamma^r_*(E)$, $r = 0, 1, \ldots,$ together. There is a description of this topology similar to (B.1) using the infinite jet bundle $J^\infty E$. As a space over M it is a limit of

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & J^r E & \longrightarrow \cdots \longrightarrow & J^1 E & \longrightarrow & E \\
& \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\cdots =\!=\!= & M & =\!=\!= \cdots =\!=\!= & M & =\!=\!= & M
\end{array}
$$

We have a similar jet prolongation map

$$j^\infty : \Gamma^\infty_*(E) \longrightarrow \Gamma^0_*(J^\infty E)$$

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and it is again an inclusion of a subspace. For a *compact* manifold M the two topologies agree $\Gamma_W^r(E) = \Gamma_S^r(E)$, as can be easily deduced from the case $r = 0$.

In the case when $E = N \times M$ is the trivial bundle (so that $\Gamma^r(E)$ is the space of all C^r-maps $M \to N$) we write $C_*^r(M, N)$ instead of $\Gamma_*^r(E)$. For the differential topology one of the most important features of these topologies is that they are all Baire spaces, i.e. countable intersections of open dense subsets are dense.

As we are only interested in the case $r = \infty$ this is what we are going to assume from now on. The following results explain how could smooth maps be defined by specifying then on subsets such that they agree on the intersections.

Lemma B.1. *Let $E \to M$ be a smooth bundle and let U be an open covering of M. Then the restriction maps $\Gamma_W^\infty(E) \to \Gamma_W^\infty(U, E)$ for $U \in \mathrm{U}$ induce a homeomorphism*

$$\Gamma_W^\infty(E) \xrightarrow{\cong} \lim_{U \in \mathrm{U}} \Gamma_W^\infty(U, E)$$

Proof. Firstly the above map is a homeomorphism when ∞ is replaced by 0, i.e. on the level of spaces of continuous sections with the compact-open topology. As for each r we have

$$\Gamma_W^0(J^r E) \xrightarrow{\cong} \lim_{U \in \mathrm{U}} \Gamma_W^0(U, J^r E)$$

and subspaces commute with limits we get a homeomorphism

$$\Gamma_W^r(E) \xrightarrow{\cong} \lim_{U \in \mathrm{U}} \Gamma_W^r(U, E)$$

by restriction. Finally, taking a limit for $r \to \infty$ gives the result (or we could take $r = \infty$ straight away). □

Lemma B.2. *If $M = M_1 \cup M_2$ is a union of two submanifolds of codimension 0 meeting at their common boundary $M_0 = M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ then the following diagram is a pullback square*

$$
\begin{array}{ccc}
\Gamma_*^\infty(E) & \longrightarrow & \Gamma_*^\infty(M_1, E) \\
\downarrow & & \downarrow \\
\Gamma_*^\infty(M_2, E) & \longrightarrow & \Gamma_*^0(M_0, J^\infty E)
\end{array}
$$

Here $\Gamma_*^\infty(M_i, E)$ denotes the space of smooth sections of E over M_i. □

Before we give some applications of this lemma we switch our attention to the case of sections of a *vector* bundle E. In this case it can be shown that the addition or subtraction of sections is continuous so that $\Gamma_*(E)$ is actually a topological abelian group. More care has to be taken with the multiplication of sections. The map

$$C_*^\infty(M, \mathbb{R}) \times \Gamma_*(E) \longrightarrow \Gamma_*(E)$$

is continuous so that $\Gamma_*(E)$ is a topological module over the topological ring $C_*^\infty(M, \mathbb{R})$. The bad news is that the inclusion $\mathbb{R} \to C_*^\infty(M, \mathbb{R})$ is continuous only for $* = W$ or when M is compact (in which case the two topologies agree anyway).

From now on we assume that M is compact and write $\Gamma(E)$ for $\Gamma_*^\infty(E)$ (we do not have to distinguish between the weak and strong topology now). As $\Gamma(E)$ is a topological

vector space we only need to understand the neighbourhood basis at 0. If one chooses a metric on each vector bundle $J^r E$ we can construct one by taking

$$\{s \in \Gamma(E) \mid \forall x \in M : ||j^r_x s|| < \varepsilon\}$$

Therefore one easily sees that $\Gamma(E)$ is a LCTVS which is defined by seminorms

$$s \mapsto \sup_{x \in M} ||j^r_x s||$$

Using Lemma B.2 we prove continuity of various maps that are used in the main chapters.

COROLLARY B.3. *Let us denote* $\mathcal{R} = C^\infty(M, \mathbb{R})$ *and* $\mathcal{R}^I = C^\infty(M \times I, \mathbb{R})$, *let* $\mathcal{S}$ *be the subspace of* $\mathcal{R}^I \times \mathcal{R}^I$ *consisting of those* (f, g) *for which there exists* $\varepsilon > 0$ *such that*

$$f(x, 1 - t) = f(x, 1) = g(x, 0) = g(x, t)$$

for all $0 \leq t \leq \varepsilon$. *The "concatenation" map*

$$\mu : (0, 1) \times \mathcal{S} \longrightarrow \mathcal{R}^I$$

sending (s, f, g) *to the function*[1]

$$h(x, t) = \begin{cases} f(x, t/s) & \text{when } 0 \leq t \leq s \\ g(x, (t - s)/(1 - s)) & \text{when } s \leq t \leq 1 \end{cases}$$

is continuous[2].

PROOF. We write $I \times (0, 1)$ as a union of two submanifolds

$$P_1 = \{(t, s) \mid s \leq t\} \qquad P_2 = \{(t, s) \mid s \geq t\}$$

of codimension 0 and define maps $\alpha_i : P_i \to I$

$$\alpha_1(t, s) = t/s \qquad \alpha_2(t, s) = (t - s)/(1 - s)$$

and therefore maps

$$\mathcal{R}^I = C^\infty(M \times I, \mathbb{R}) \xrightarrow{(\mathrm{id} \times \alpha_i)^*} C^\infty_W(M \times P_i, \mathbb{R})$$

(for the continuity of this map see the first paragraph after the proof). Taking a product of these maps we obtain the bottom map in the diagram

$$\mathcal{S} \dashrightarrow^{\beta} C^\infty_W(M \times I \times (0, 1), \mathbb{R})$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\mathcal{R}^I \times \mathcal{R}^I \longrightarrow C^\infty_W(M \times P_1, \mathbb{R}) \times C^\infty_W(M \times P_2, \mathbb{R})$$

According to Lemma B.2 the dashed arrow is continuous. We can write the concatenation map μ as a composition

$$(0, 1) \times \mathcal{S} \xrightarrow{\mathrm{id} \times \beta} (0, 1) \times C^\infty_W(M \times I \times (0, 1), \mathbb{R}) \xrightarrow{pev} C^\infty(M \times I, \mathbb{R}) = \mathcal{R}^I$$

with *pev* being the partial evaluation $(s, f) \mapsto f(-, s)$. $\square$

---

[1] Geometrically just f reparametrized to $M \times [0, s]$ patched with g reparametrized to $M \times [s, 1]$.

[2] This is true even on a bigger subspace of those (f, g) whose infinite jets are compatible.

The composition $C^\infty(N, P) \times C^\infty(M, N) \longrightarrow C^\infty(M, P)$ is continuous in the weak topologies and for strong topologies it is continuous on the subspace formed by (f, g) with g proper. This proves most of

LEMMA B.4. *Let $\lambda : I \to I$ be a smooth function. The reparametrization map*

$$\lambda^* : \mathcal{R}^I \to \mathcal{R}^I$$

defined by $f(x, t) \mapsto f(x, \lambda(t))$ is continuous. Moreover for a smooth family $\lambda_\tau : I \to I$ the induced homotopy

$$\lambda_\tau^* : \mathcal{R}^I \to \mathcal{R}^I$$

is also continuous.

PROOF. Again the homotopy can be written as

$$I \times C^\infty(M \times I, \mathbb{R}) \xrightarrow{(\mathrm{id} \times \lambda)^*} I \times C^\infty(M \times I \times I, \mathbb{R}) \xrightarrow{pev} C^\infty(M \times I, \mathbb{R})$$

with $\lambda : I \times I \to I$ sending (t, τ) to $\lambda_\tau(t)$. $\qquad\square$

COROLLARY B.5. *Let $0 < s < 1$ and $\lambda : I \to I$, $\rho : I \to I$ two functions such that $\lambda(0) = 0$, $\lambda = 1$ near 1 and similarly $\rho(1) = 1$, $\rho = 0$ near 0. The concatenation map*

$$\mu(s, \lambda, \rho) : \mathcal{R}^I \times_\mathcal{R} \mathcal{R}^I \longrightarrow \mathcal{R}^I$$

given by $(f, g) \mapsto \mu(s, \lambda^* f, \rho^* g)$ is continuous. Moreover for any other choice of s, λ and ρ the resulting map is homotopic via maps of the same form.

PROOF. Any two values of s can be joined by a linear homotopy as can be any two values of λ and ρ. $\qquad\square$

Sheaves

The main references for this appendix are the books [**Bre**] and [**GM**]. The idea of this appendix is to gather facts about sheaves and in particular about sheaf cohomology that are used in the thesis (mainly in Appendix D and in Chapter 4).

The basic intuition behind sheaves is that they are systems of abelian groups parametrized by the points of a topological space X. In one approach they indeed are certain spaces over X. In this sense they generalize the so-called systems of local coefficients (also known as bundles of abelian groups). Moreover they constitute an abelian category on which a variety of functors is defined. For example, we have a functor of global sections which associates to a sheaf the abelian group of its sections. Its derived functors are the sheaf cohomology groups with coefficients in the sheaf under consideration. This is roughly the content of the first section.

There are other cohomology theories with coefficients in sheaves (or in a special class of sheaves) and among them the sheaf cohomology is "universal". For paracompact spaces most of these theories agree. One exception is the singular cohomology which does not behave well for spaces with local complexities. So far for the section 2.

In the last section we pursue more advanced properties of sheaves and in particular derive the Leray spectral sequence. Given a continuous map $f : X \to Y$ it relates the sheaf cohomology of X to the sheaf cohomology of Y and of the fibres $f^{-1}(y)$ over the points $y \in Y$.

C.1. Sheaves, sheaf cohomology

Let X be a topological space[1]. We denote by $\mathrm{Op}(X)$ the poset of all open subsets of X ordered by inclusion. A *presheaf* on X is just a functor $\mathrm{Op}(X)^{op} \to \mathsf{Set}$ to the category of sets. Therefore we have a category of presheaves on X, namely the functor category $\mathsf{Pr}_X\mathsf{Set} = \mathsf{Set}^{\mathrm{Op(X)^{op}}}$. In a similar way we define a presheaf in any category $\mathcal{C}$ as an object of $\mathsf{Pr}_X\mathcal{C} = \mathcal{C}^{\mathrm{Op(X)^{op}}}$. A presheaf $\mathcal{A} \in \mathsf{Pr}_X\mathcal{C}$ is said to be a *sheaf* if for every subset $\mathrm{U} \subseteq \mathrm{Op}(X)$ which is closed under subsets the map

$$\mathcal{A}(\cup \mathrm{U}) \longrightarrow \varprojlim_{U \in \mathrm{U}} \mathcal{A}(U)$$

(with $\cup \mathrm{U} = \bigcup_{U \in \mathrm{U}} U$) is an isomorphism, i.e. if $\mathcal{A}$ preserves limits over all hereditary subsets. In this way we get a full subcategory $\mathsf{Sh}_X\mathcal{C}$ of sheafs of objects in $\mathcal{C}$ on X.

---

[1]we do not assume here that it is compactly generated Hausdorff

Suppose that $\mathcal{C}$ is an "algebraic category" - a category $\langle \Omega, E \rangle$-$\mathsf{Alg}$ of algebras[2] defined by operations Ω and relations E. Then the inclusion $\iota : \mathsf{Sh}_X\mathcal{C} \hookrightarrow \mathsf{Pr}_X\mathcal{C}$ has a left adjoint $s : \mathsf{Pr}_X\mathcal{C} \to \mathsf{Sh}_X\mathcal{C}$ sometimes called sheafification. It is given by the formula

$$s\mathcal{A}(U) = \operatorname*{colim}_{V \in \mathrm{Cov}_h(U)} \lim_{V \in \mathrm{V}} \mathcal{A}(V)$$

where the colimit is taken over all hereditary open coverings of U.

A more illuminating description of the sheafification functor can be obtained by introducing total spaces of presheaves. Let $\mathcal{A} \in \mathsf{Pr}_X\mathsf{Set}$ be a presheaf of sets on X. We construct a space $\pi_\mathcal{A} : \mathrm{Tot}(\mathcal{A}) \to X$ over X called the *total space* of $\mathcal{A}$

$$\pi_\mathcal{A} : \mathrm{Tot}(\mathcal{A}) = \int^{U \in \mathrm{Op}(X)} U \times \mathcal{A}(U) \longrightarrow \int^{U \in \mathrm{Op}(X)} U = X$$

The projection $\pi_\mathcal{A}$ can be shown to be a local homeomorphism and there is a bijection between $(s\mathcal{A})(U)$ and the set $\Gamma(U, \mathrm{Tot}(\mathcal{A}))$ of sections of $\mathrm{Tot}(\mathcal{A})$ over U. In this way the category $\mathsf{Sh}_X\mathsf{Set}$ of sheaves of sets on X becomes equivalent to the category of local homeomorphisms $A \to X$. Moreover Tot is identified with the sheafification functor s and the functor $\Gamma(-, A)$ of local sections with the inclusion functor ι.

For the category $\mathcal{C} = \langle \Omega, E \rangle$-$\mathsf{Alg}$ we get an equivalence of $\mathsf{Sh}_X\mathcal{C}$ with the category of $\langle \Omega, E \rangle$-algebras in $\mathsf{Sh}_X\mathsf{Set}$, i.e. the category of local homeomorphisms $A \to X$ with a fibrewise structure of an $\langle \Omega, E \rangle$-algebra. Of course covering maps are local homeomorphisms and therefore they give examples of sheaves, the so-called *locally constant* sheaves. Also systems of local coefficients (or bundles of groups) on X just correspond to locally constant sheaves of abelian groups. The fibres $\mathcal{A}_x := \pi_\mathcal{A}^{-1}(x)$ of the projection $\pi_\mathcal{A}$ are called the *stalks* of the sheaf $\mathcal{A}$. Alternatively one can define them as the directed colimits

$$\mathcal{A}_x = \operatorname*{colim}_{\substack{U \in \mathrm{Op}(X) \\ x \in U}} \mathcal{A}(U)$$

Now assume that in addition $\mathcal{C} = \langle \Omega, E \rangle$-$\mathsf{Alg}$ is an abelian category, main example being the category of abelian groups. Then both $\mathsf{Pr}_X\mathcal{C}$ and $\mathsf{Sh}_X\mathcal{C}$ become also abelian with ι left exact and s exact. Moreover if $\mathcal{C}$ has enough injectives then so do both $\mathsf{Pr}_X\mathcal{C}$ and $\mathsf{Sh}_X\mathcal{C}$ and if $\mathcal{C}$ has enough projectives then the same is true for $\mathsf{Pr}_X\mathcal{C}$ *but not* for $\mathsf{Sh}_X\mathcal{C}$ in general. From now on we assume that $\mathcal{C}$ and hence also $\mathsf{Sh}_X\mathcal{C}$ have enough injectives.

The abelian structure of the category $\mathsf{Sh}_X\mathcal{C}$ is reflected in the stalks, namely a sequence of sheaves is exact if and only if at each point the corresponding sequence of stalks is exact. This is where the geometric description of a sheaf (in terms of the total space) is very useful.

It is also this description that gives a name to the elements of $\mathcal{A}(U)$ - they are called sections of $\mathcal{A}$ over U. Hence we often write $\Gamma(U, \mathcal{A})$ instead of $\mathcal{A}(U)$ and for global sections simply $\Gamma(\mathcal{A})$. In this way we get a "global sections" functor $\Gamma : \mathsf{Sh}_X\mathcal{C} \to \mathcal{C}$.

---

[2]Here an $\langle \Omega, E \rangle$-algebra is a set S together with a realization of Ω by operations $S^n \to S$ that satisfy the relations E. The category $\mathsf{Sh}_X \langle \Omega, E \rangle$-$\mathsf{Alg}$ can be alternatively described as the category of $\langle \Omega, E \rangle$-algebras in the category $\mathsf{Sh}_X\mathsf{Set}$ of sheaves of sets.

The essence of the sheaf theory is that this functor is not exact but merely left exact. As the category $\mathsf{Sh}_X\mathcal{C}$ has enough injectives one can define the right derived functors of Γ. In this appendix we denote $R^n\Gamma(\mathcal{A}) \in \mathcal{C}$ by $H^n(X;\mathcal{A})$ and call it the n-th *sheaf cohomology* of X with coefficients in the sheaf $\mathcal{A}$. If φ is a family of supports on X we can define $\Gamma_\varphi(\mathcal{A})$, the sections of $\mathcal{A}$ with supports in φ, to be the kernel of

$$\Gamma(\mathcal{A}) \longrightarrow \operatorname*{colim}_{F \in \varphi} \Gamma(X - F; \mathcal{A})$$

Again Γ_φ is left exact and there are right derived functors $R^n\Gamma_\varphi(\mathcal{A}) = H^n_\varphi(X;\mathcal{A})$, the n-th sheaf cohomology of X with supports φ and with coefficients in $\mathcal{A}$.

C.2. The relation to other cohomology theories

In this section we relate the sheaf cohomology to the singular cohomology H^*_Δ and to the Čech cohomology $\check{H}^*$.

First we treat the singular cohomology with constant coefficients and any supports. For an open subset $U \in \mathrm{Op}(X)$ let us condiser the singular cochain complex $C^*(U;\mathbb{Z})$ on U. It provides a cochain complex of presheaves $C^*(-;\mathbb{Z})$ of abelian groups on X. By definition

$$H^*_\Delta(X) = H^*(C^*(X;\mathbb{Z}))$$

We denote by $\mathcal{S}^* \in \mathsf{Sh}_X\mathsf{Ab}$ the cochain complex of generated sheaves, i.e.

$$\mathcal{S}^* = sC^*(-;\mathbb{Z})$$

It turns out (see Section I.7 of [**Bre**]) that if X is paracompact then

$$H^*_\Delta(X) \cong H^*(\mathcal{S}^*(X)) = H^*(\Gamma(\mathcal{S}^*))$$

We can define more generally the singular cohomology groups of X with supports in φ and coefficients in a locally constant sheaf $\mathcal{A}$ as the cohomology of the cochain complex

$$C^p_\varphi(X;\mathcal{A}) = \{\alpha : \mathsf{map}(\Delta^p, X) \to \mathsf{map}(\Delta^p, \mathrm{Tot}(\mathcal{A})) \mid \pi_*\alpha = \mathrm{id}, \ \mathrm{supp}\,\alpha \in \varphi\}$$

of presheaves of abelian groups. If the family φ is paracompactifying we get again an isomorphism

$$H^*_{\Delta,\varphi}(X;\mathcal{A}) \cong H^*(\Gamma_\varphi(\mathcal{S}^*_\mathcal{A}))$$

where $\mathcal{S}^*_\mathcal{A}$ is the sheaf generated by the presheaf $C^*(-;\mathcal{A})$. Under the same assumptions the sheaves $\mathcal{S}^*_\mathcal{A}$ (assuming that $\mathcal{A}$ is finitely generated, see Section III.1 of [**Bre**]) turn out to be Γ_φ-acyclic so that $H^*_{\Delta,\varphi}$ computes the sheaf cohomology of X with supports in φ and with coefficients in $\mathcal{A}$ provided that the cochain complex

$$\cdots \to 0 \to \mathcal{A} \to \mathcal{S}^0_\mathcal{A} \to \mathcal{S}^1_\mathcal{A} \to \cdots$$

is acyclic. The stalk at x of its n-th cohomology is clearly

$$\operatorname*{colim}_{\substack{U \in \mathrm{Op}(X) \\ x \in U}} \tilde{H}^*_\Delta(U; \mathcal{A}|U)$$

Therefore for a paracompactifying family φ the natural map $H^*_{\Delta,\varphi}(X;\mathcal{A}) \to H^*_\varphi(X;\mathcal{A})$ is an isomorphism if this colimit vanishes or in other words if all the points $x \in X$ are taut with respect to the singular cohomology. This is for example the case for all

locally contractible spaces X. Also it is implied by the condition HLC - homological local connectedness.

In general there is a convergent spectral sequence

$$E_2^{pq} = H_\varphi^p(X; sH_\Delta^q(-; \mathcal{A})) \Rightarrow H_{\Delta,\varphi}^{p+q}(X; \mathcal{A})$$

Next comes the Alexander-Spanier cohomology. There is a sequence of abelian groups

$$AS^p(U; \mathbb{Z}) = \mathsf{map}(U^{p+1}, \mathbb{Z})$$

It is easy to define the coface and codegeneracy maps making it into a cosimplicial presheaf of abelian groups. The associated cochain complex will be denoted again by $AS^*(-; \mathbb{Z})$. Let $\mathcal{AS}^*$ be the sheaf generated by this presheaf. Then by definition

$$H_{AS,\varphi}^* = H^*(\Gamma_\varphi(\mathcal{AS}^*))$$

Again if φ is paracompactifying then all the sheaves $\mathcal{AS}^*$ are Γ_φ-acyclic and $\check{H}_\varphi^*$ is isomorphic to the corresponding sheaf cohomology provided that $\mathcal{AS}^*$ constitutes a resolution of $\mathbb{Z}$. Unlike in the case of singular cohomology this is *always* the case.

Let $\mathcal{A} \in \mathsf{Pr}_X\mathsf{Ab}$ be a presheaf and $U \subseteq \mathsf{Op}(X)$ any open covering of X. We define

$$\check{C}^p(U; \mathcal{A}) = \prod_{(U_0,\ldots,U_p)\in U^{p+1}} \mathcal{A}(U_0 \cap \cdots \cap U_p)$$

Again one can make $\check{C}^*(U; \mathcal{A})$ into a cosimplicial abelian group and further into a cochain complex. We define the Čech cohomology of the covering U with coefficients in $\mathcal{A}$ as

$$\check{H}^*(U; \mathcal{A}) = H^*(\check{C}^*(U; \mathcal{A}))$$

If V is a refinement of U and $\lambda : V \to U$ is a function such that $V \subseteq \lambda V$ then we define a map

$$\lambda^* : \check{C}^p(U; \mathcal{A}) \longrightarrow \check{C}^p(V; \mathcal{A})$$

using the restriction maps as in the diagram

$$
\begin{array}{ccc}
\check{C}^p(U; \mathcal{A}) & \dashrightarrow & \check{C}^p(V; \mathcal{A}) \\
\downarrow & & \downarrow \\
\mathcal{A}(\lambda V_0 \cap \cdots \cap \lambda V_p) & \xrightarrow{res} & \mathcal{A}(V_0 \cap \cdots \cap V_p)
\end{array}
$$

It is a map of cosimplicial abelian groups and therefore induces a map

$$\lambda^* : \check{H}^*(U; \mathcal{A}) \longrightarrow \check{H}^*(V; \mathcal{A})$$

on cohomology. For a different choice of λ one can show that the resulting map of cosimplicial abelian groups is homotopic to λ^* and so on the level of cohomology one gets an equality. Hence we can define

$$\check{H}^*(X; \mathcal{A}) = \operatorname*{colim}_{U \in \mathsf{Cov}(X)} \check{H}^*(U; \mathcal{A})$$

where the colimit is taken over all open coverings of X ordered by refinement. This group is called the Čech cohomology of X with coefficients in the presheaf $\mathcal{A}$.

Consider the following composition

$$\Gamma : \mathsf{Sh}_X\mathsf{Ab} \xrightarrow{\check{C}^*(\mathrm{U};-)} \mathsf{Ch}^{\geq 0}\mathsf{Ab} \xrightarrow{H^0} \mathsf{Ab}$$

As the derived functors are easily seen to be $R^p H^0 \cong H^p$ and $(R^q\check{C}^*(\mathrm{U};-))(\mathcal{A}) \cong \check{C}^*(\mathrm{U};H^q(-;\mathcal{A}))$ the Grothendieck spectral sequence takes the form

$$E_2^{pq}(\mathrm{U}) \cong \check{H}^p(\mathrm{U};H^q(-;\mathcal{A})) \Rightarrow H^{p+q}(X;\mathcal{A})$$

Taking a colimit over the open coverings U of X one gets again a convergent spectral sequence

$$E_2^{pq} \cong \check{H}^p(X;H^q(-;\mathcal{A})) \Rightarrow H^{p+q}(X;\mathcal{A})$$

According to Spanier (with an additional assumption of X being paracompact) for any presheaf $\mathcal{B} \in \mathsf{Pr}_X\mathsf{Ab}$ we have an isomorpism

$$\check{H}^*(X;\mathcal{B}) \xrightarrow{\cong} \check{H}^*(X;s\mathcal{B})$$

As clearly $sH^q(-;\mathcal{A}) = 0$ for $q > 0$ and $sH^0(-;\mathcal{A}) = \mathcal{A}$ the colimit spectral sequence collapses to give an isomorphism[3]

$$\check{H}^*(X;\mathcal{A}) \xrightarrow{\cong} H^*(X;\mathcal{A})$$

C.3. The sheaf cohomology and continuous maps

Now we describe how sheaves behave with respect to continuous maps. If $f : X \to Y$ is a continuous map then we get an induced functor $f^* : \mathrm{Op}(Y) \to \mathrm{Op}(X)$, given simply by $f^*U = f^{-1}(U)$, and therefore a functor between the presheaf categories called the *direct image*

$$f_\bullet : \mathsf{Pr}_X\mathcal{C} \to \mathsf{Pr}_Y\mathcal{C}$$

It is easy to check that this functor preserves the subcategory of sheaves and we have

$$f_\bullet : \mathsf{Sh}_X\mathcal{C} \to \mathsf{Sh}_Y\mathcal{C}$$

Explicitly $f_\bullet\mathcal{A}(U) = \mathcal{A}(f^{-1}(U))$. The functor $f_\bullet : \mathsf{Pr}_X\mathcal{C} \to \mathsf{Pr}_Y\mathcal{C}$ has a left adjoint Lan_{f^*}, the left Kan extension along f^*. Therefore we have a composition

$$f^\bullet : \mathsf{Sh}_Y\mathcal{C} \xrightarrow{\iota} \mathsf{Pr}_Y\mathcal{C} \xrightarrow{\mathrm{Lan}_{f^*}} \mathsf{Pr}_X\mathcal{C} \xrightarrow{s} \mathsf{Sh}_X\mathcal{C}$$

It is called the *inverse image* and we claim that it is a left adjoint of $f_\bullet$. This is clear from

$$\mathsf{Sh}_X\mathcal{C}(f^\bullet\mathcal{A},\mathcal{B}) \cong \mathsf{Pr}_X\mathcal{C}(\mathrm{Lan}_{f^*}\iota\mathcal{A},\iota\mathcal{B}) \cong \mathsf{Pr}_Y\mathcal{C}(\iota\mathcal{A},f_\bullet\iota\mathcal{B}) \cong \mathsf{Sh}_Y\mathcal{C}(\mathcal{A},f_\bullet\mathcal{B})$$

A very nice description of the functor $f^\bullet$ is in terms of the total spaces. It is plainly a pullback functor along the map f, sending $A \to Y$ to $A \times_Y X \to X$.

$$
\begin{array}{ccc}
A \times_Y X & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

---

[3]even when X is not paracompact this map is an isomorphism for $* = 0,1$ and a monomorphism for $* = 2$.

In particular $(f^\bullet\mathcal{A})_x \cong \mathcal{A}_{f(x)}$ and therefore the functor $f^\bullet$ is exact. On the other hand $f_\bullet$ is merely left exact and thus one can consider its right derived functors called the *higher direct images*. Moreover as $f^\bullet$ is exact, $f_\bullet$ preserves injectives and one gets a Grothendieck spectral sequence for the composition $\Gamma(Y, f_\bullet\mathcal{A}) = \Gamma(X, \mathcal{A})$

$$E_2^{pq} = H^p(Y; R^q f_\bullet(\mathcal{A})) \Rightarrow H^{p+q}(X; \mathcal{A})$$

In this context it is known as the Leray spectral sequence. In order to use this spectral sequence one has to have at least some understanding of the higher direct images $R^n f_\bullet$. Clearly $R^0 f_\bullet = f_\bullet$ and under some conditions one can say something at least about the stalks $(R^n f_\bullet(\mathcal{A}))_y$. Let $\mathcal{A} \to \mathcal{I}^*$ be an injective resolution of $\mathcal{A}$ in $\mathsf{Sh}_X\mathcal{C}$. Then $(R^n f_\bullet(\mathcal{A}))_y$ is by definition

$$(H^n(f_\bullet\mathcal{I}^*))_y \cong H^n((f_\bullet\mathcal{I}^*)_y) \cong H^n\left(\operatorname*{colim}_{\substack{U\in\mathrm{Op}(Y)\\y\in Y}} \mathcal{I}^*(f^*U)\right) \cong \operatorname*{colim}_{\substack{U\in\mathrm{Op}(Y)\\y\in Y}} H^n(\mathcal{I}^*(f^*U))$$

and moreover $H^n(\mathcal{I}^*(f^*U)) \cong H^n(f^*U; \mathcal{A}|f^*U)$. If f is closed (i.e. the image under f of a closed subset is also closed) and the fibres $f^{-1}(y)$ over points y of Y are "taut" then one has an isomorphism (proved as Proposition IV.4.2. in [**Bre**])

$$\operatorname*{colim}_{\substack{U\in\mathrm{Op}(Y)\\y\in Y}} H^n(f^*U; \mathcal{A}|f^*U) \cong H^n(f^{-1}(y); \mathcal{A}|f^{-1}(y))$$

THEOREM C.1. *Suppose that $f : X \to Y$ is a continuous map. Then there is a convergent spectral sequence*

$$E_2^{pq} = H^p(Y; R^q f_\bullet(\mathcal{A})) \Rightarrow H^{p+q}(X; \mathcal{A})$$

If moreover f is closed and each $f^{-1}(y)$ taut then there is an isomorphism

$$(R^q f_\bullet(\mathcal{A}))_y \cong H^n(f^{-1}(y); \mathcal{A}|f^{-1}(y))$$

Of course, the stalks of a sheaf do not describe the sheaf completely unless, for example, they are all zero. In this special case we get the Vietoris-Begle mapping theorem

THEOREM C.2. *Suppose that $f : X \to Y$ is a closed continuous map such that each $f^{-1}(y)$ is taut and suppose that $H^n(f^{-1}(y); \mathcal{A}|f^{-1}(y)) = 0$ for all $y \in Y$ and for all $n > 0$. Then there is an isomorphism*

$$H^*(Y; f_\bullet\mathcal{A}) \cong H^*(X; \mathcal{A})$$

Now we will sketch a generalization for the sheaf cohomology with supports. Let ψ be a family of supports on X. There is a "direct image with supports in ψ" functor

$$f_\psi : \mathsf{Sh}_X\mathcal{C} \to \mathsf{Sh}_Y\mathcal{C}$$

Here $f_\psi\mathcal{A}$ is defined as the sheafification of the presheaf

$$U \mapsto \Gamma_{\psi \cap f^{-1}U}(f^{-1}U, \mathcal{A})$$

and f_ψ is easily seen to be left exact. If φ is a family of supports on Y then there is another family called $\varphi(\psi)$ on Y such that

$$\Gamma_\varphi(f_\psi\mathcal{A}) = \Gamma_{\varphi(\psi)}(\mathcal{A})$$

Under the additional hypothesis that φ is paracompactifying we get (in the same way but with a lot more work caused by f_ψ not preserving injectives) a convergent spectral sequence

$$E_2^{pq} = H_\varphi^p(Y; R^q f_\psi(\mathcal{A})) \Rightarrow H_{\varphi(\psi)}^{p+q}(X; \mathcal{A})$$

If for every $F \in \psi$ the image $f(F)$ is closed (i.e. if f is ψ-closed) and if each $f^{-1}(y)$ is ψ-taut then we can identify the stalks as

$$(R^n f_\psi(\mathcal{A}))_y \cong H_\psi^n(f^{-1}(y); \mathcal{A}|f^{-1}(y))$$

We are particularly interested in the case where X and Y are locally compact Hausdorff spaces and both φ and ψ are the families of compact supports. In this case $\varphi(\psi)$ is also the family c of compact supports and clearly every continuous f is c-closed. Also every closed subset of X is c-taut and we have the following results

THEOREM C.3. *Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Then there is a convergent spectral sequence*

$$E_2^{pq} = H_c^p(Y; R^q f_c(\mathcal{A})) \Rightarrow H_c^{p+q}(X; \mathcal{A})$$

Moreover $(R^q f_c(\mathcal{A}))_y \cong H_c^q(f^{-1}(y); \mathcal{A}|f^{-1}(y))$.

THEOREM C.4. *Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces such that $H_c^n(f^{-1}(y); \mathcal{A}|f^{-1}(y)) = 0$ for all $y \in Y$ and for all $n > 0$. Then there is an isomorphism*

$$H_c^*(Y; f_c\mathcal{A}) \cong H_c^*(X; \mathcal{A})$$

If moreover f is proper and each $f^{-1}(y)$ connected then $f = f_c$ and the canonical map $\mathcal{B} \xrightarrow{\cong} f_\bullet f^\bullet \mathcal{B}$ is an isomorphism. Consequently for an any $\mathcal{B} \in \mathsf{Sh}_Y \mathcal{C}$

$$H_c^*(Y; \mathcal{B}) \cong H_c^*(X; f^\bullet \mathcal{B})$$

APPENDIX D

Alexander duality, transfer maps

This appendix is concerned with Alexander duality for oriented topological manifolds. If $f : M \to N$ is a continuous map between such manifolds and if $X \subseteq M$ and $Y \subseteq N$ are closed subsets such that $f^{-1}(Y) \subseteq X$ then we have the following diagram

$$
\begin{array}{ccc}
H_*(M, M - X) & \xrightarrow{\;f_*\;} & H_*(N, N - Y) \\
\cong \downarrow & & \downarrow \cong \\
\check{H}_c^{m-*}(X) & \longrightarrow & \check{H}_c^{n-*}(Y)
\end{array}
$$

and a natural question arises: is there a reasonable description of the bottom map? We give a positive answer for the case of inclusion of a zero section of a vector bundle (which easily leads to the answer in the case of an embedding of a smooth submanifold). This map is "locally" a cup product with a Thom class and in this sense generalizes the Thom isomorphism. We start with a general discussion of singular (co)homology with general supports and at the end prove the promised Theorem D.3.

Let X be a topological space. We say that a family φ of closed subsets of X is a family of supports on X if φ is closed under finite unions and taking closed subsets. We say that φ is paracompactifying if all elements of φ are paracompact and if any $F \in \varphi$ has a neighbourhood $F \subseteq F'$ such that $F' \in \varphi$.

For any space X we denote by ϕ the family of all closed subsets and by c the family of all compact subsets. Note that ϕ is paracompactifying if X is paracompact and c is paracompactifying if X is locally compact Hausdorff.

If $A \subseteq X$ is a subspace and φ a family of supports on X we denote by $\varphi|_A$ the family

$$
\varphi|_A = \{F \in \varphi \mid F \subseteq A\}
$$

It is a family of supports on A. Another useful family of supports on A is

$$
\varphi \cap A = \{F \cap A \mid F \in \varphi\}
$$

Note that if A is closed $\varphi|_A = \varphi \cap A$.

We define the (singular) cohomology of X with supports in φ as

$$
H_\varphi^*(X) = \operatorname*{colim}_{F \in \varphi} H^*(X, X - F)
$$

The dual construction for homology yields

$$
H_*^\varphi(X) = \lim_{F \in \varphi} H_*(X, X - F)
$$

For an open subset $U \subseteq X$ we also define the relative versions

$$H_\varphi^*(X, U) = H_{\varphi|X-U}^*(X) \qquad \text{and} \qquad H_*^\varphi(X, U) = H_*^{\varphi|X-U}(X)$$

Note that although the above cohomology groups deserve to be called "with supports in φ", it is not the case for homology. Obviously $H_* = H_*^\phi$ (resp. $H^* = H_\phi^*$) are isomorphic to the singular homology (resp. cohomology) groups while H_c^* is the usual cohomology with compact supports and H_*^c is the locally finite homology (whereas the singular homology H_* *does* have compact supports).

Remark. OK, it is actually not true. The right thing to do (at least in the case of locally compact spaces) is to take the above limit on the level of chain complexes and then take homology. If the space is moreover paracompact then one has a split short exact sequence

$$0 \to \lim_{F \in \varphi}^1 (H_{*+1}(X, X - F)) \to H_*(\lim_{F \in \varphi} C_*(X, X - F)) \to \lim_{F \in \varphi} H_*(X, X - F) \to 0$$

Thus the group that we have defined is a quotient (unnaturally a direct summand) of the usual one. In what proceeds we only use this H_*^φ group in the case of a top dimensional locally finite homology of an orientable m-dimensional manifold M where we have

$$0 \to \lim_{F \in c}^1 (H_{m+1}(M, M - F)) \to H_m^{lf}(M) \to H_m^c(M) \to 0$$

and the $\lim^1$ term vanishes trivially so that the two homology groups are naturally isomorphic.

If $f : X \to Y$ is a continuous map and ψ is a family of supports on Y we get the induced family of supports $f^{-1}\psi$ on X as

$$f^{-1}\psi = \{F \subseteq X \mid \overline{fF} \in \psi\}$$

the closure of $\{f^{-1}F \mid F \in \psi\}$ under taking closed subsets. Note that if $f : X \subseteq Y$ is an inclusion of a subspace then $f^{-1}\psi = \psi \cap A$.

Obviously if $f^{-1}\psi \subseteq \varphi$ for some family φ of supports on X (or in other words if $F \in \psi \Rightarrow f^{-1}F \in \varphi$, an obvious generalization of a proper map) we get induced maps

$$f^* : H_\psi^*(Y) \to H_\varphi^*(X)$$
$$f_* : H_*^\varphi(X) \to H_*^\psi(Y)$$

Suppose that φ is a paracompactifying family of supports on X and $A \subseteq X$ a closed subset. Then the inclusion induces a map $H_\varphi^*(X) \to H_{\varphi|A}^*(A)$. If U is an open subset of X then we obtain a map

$$H_{\varphi|U}^*(U) = \operatorname*{colim}_{F \in \varphi|U} H^*(U, U-F) \cong \operatorname*{colim}_{F \in \varphi|U} H^*(X, X-F) \to \operatorname*{colim}_{F \in \varphi} H^*(X, X-F) = H_\varphi^*(X)$$

going in the opposite direction. The isomorphism comes from excision. This is where we use the assumption on φ.

Now let φ and ψ be two families of supports on X. Then we have cup products

$$\cup : H_\varphi^*(X) \otimes H_\psi^*(X) \to H_{\varphi \cap \psi}^*(X)$$

and cap products
$$\cap : H^*_\varphi(X) \otimes H^{\varphi \cap \psi}_*(X) \to H^\psi_*(X)$$
They are defined as (co)limits of the corresponding products in singular (co)homology.

A useful formulation of the Thom isomorphism is the following. If $p : E \to B$ is an n-dimensional oriented vector bundle there is an orientation class $\tau \in H^*_{c_p}(E)$ such that the map
$$- \cup \tau : H^*(E) \to H^{*+n}_{c_p}(E)$$
is an isomorphism, where c_p denotes the family of closed subsets $C \subseteq E$ for which the restricted projection $p|_C : C \to B$ is proper. Obviously if B is compact then $c_p = c$ and we get an isomorphism (with a shift) between ordinary cohomology and cohomology with compact supports.

An immediate generalization is the following. Let φ be a family of supports on B. Then
$$- \cup \tau : H^*_{p^{-1}\varphi}(E) \to H^{*+n}_{(p^{-1}\varphi) \cap c_p}(E)$$
is an isomorphism. Also the projection induces an isomorphism
$$p^* : H^*_\varphi(B) \to H^*_{p^{-1}\varphi}(E)$$
whose inverse is induced by the inclusion $i : B \to E$ of B as a zero section of E (for reasons that will become clear later we prefer to use this inclusion instead of the projection map). If one takes φ to be c, then easily $(p^{-1}c) \cap c_p = c$ and we get a Thom isomorphism
$$H^*_c(B) \xleftarrow[i^*]{\cong} H^*_{p^{-1}c}(E) \xrightarrow[-\cup\tau]{\cong} H^{*+n}_c(E)$$
which is a model for our next proposition.

Dually we have a Thom isomorphism in homology
$$\tau \cap - : H^{(p^{-1}\varphi) \cap c_p}_{*+n}(E) \to H^{p^{-1}\varphi}_*(E)$$
and again as a special case when $\varphi = c$
$$H^c_{*+n}(E) \xrightarrow[\tau\cap-]{\cong} H^{p^{-1}c}_*(E) \xleftarrow[i_*]{\cong} H^c_*(B)$$

If moreover both E and B are oriented manifolds then their fundamental classes have to correspond under this isomorphism, at least up to a sign. One can verify that they correspond precisely (not just up to a sign) if the orientation of the fibres of p is chosen in such a way that the (local) trivialization $E \cong \mathbb{R}^n \times B$ preserves orientations (with the fibre on the right we would have to introduce a sign).

All of the above applies equally well to the Čech cohomology groups: let $X \subseteq M$ be a subspace of a (paracompact) topological manifold M. It is well-known that
$$\check{H}^*(X) \cong \operatorname*{colim}_U H^*(U)$$
where U ranges over any cofinal system of neighbourhoods of X in M. Let φ be a paracompactifying family of supports on M. Then for the Čech cohomology with supports in φ
$$\check{H}^*_{\varphi \cap X}(X) \cong \operatorname*{colim}_{F \in \varphi} \check{H}^*(X, X - F)$$

we have similarly[1]

$$\check{H}^*_{\varphi \cap X}(X) \cong \operatorname*{colim}_{U} H^*_{\varphi \cap U}(U)$$

with U ranging over any cofinal system of neigbourhoods of X. If X is closed then the closed neighbourhoods of X form such a cofinal system and in this case

$$\check{H}^*_{\varphi|_X}(X) \cong \operatorname*{colim}_{U} H^*_{\varphi|_U}(U)$$

In particular

$$\check{H}^*_c(X) \cong \operatorname*{colim}_{U} H^*_c(U)$$

Here it is important to restrict to closed neighbourhoods (otherwise the maps in this system would not have been even defined).

PROPOSITION D.1. *Let $p : E^n \to B^m$ be a vector bundle where both E and B are oriented (paracompact) topological manifolds and let $j : X \subseteq E$ be any closed subset. If we think of B as a submanifold of E via the zero section $i : B \to E$ then the following diagram commutes*

$$
\begin{array}{ccc}
H_*(B, B - X) & \xrightarrow{\quad\quad i_* \quad\quad} & H_*(E, E - X) \\
{\scriptstyle -\cap o_B} \uparrow {\scriptstyle \cong} & {\scriptstyle -\cap i_* o_B} \nearrow & {\scriptstyle \cong} \uparrow {\scriptstyle -\cap o_E} \\
\check{H}^{m-*}_c(B \cap X) \xleftarrow{\ k^* \ } & \check{H}^{m-*}_{p^{-1}c}(X) \xrightarrow{\ -\cup j^*\tau \ } & \check{H}^{n-*}_c(X)
\end{array}
$$

where $\tau \in H^{n-m}_{c_p}(E)$ is the Thom class of p and $o_E \in H^c_n(E)$, $o_B \in H^c_m(B)$ are the fundamental classes. The maps denoted by cap products are the Alexander duality maps and will be described more closely in the proof.

PROOF. The map denoted by $- \cap o_E$ in the diagram should be interpreted as follows. If U is any closed neighborhood of X then o_E gives rise to a class in

$$H^c_n(E, E - X) \cong H^c_n(U, U - X)$$

Let us call this class ξ_E. Hence, using the cap product

$$H^{n-*}_c(U) \otimes H^c_n(U, U - X) \xrightarrow{\ \cap \ } H_*(U, U - X)$$

we get a homomorphism

$$H^{n-*}_c(U) \xrightarrow{\ -\cap\xi_E \ } H_*(U, U - X) \to H_*(E, E - X)$$

---

[1] This isomorphism follows from the following facts: if φ is paracompactifying, then it is also paracompactifying for the pair (M, X) (see Definition II.9.14. of [**Bre**]; the additional property that any $F \cap X \in \varphi \cap X$ has a fundamental system of paracompact neighbourhoods holds as in a metrizable space M every subset is paracompact). This implies (Proposition II.9.15. of [**Bre**]) that X is φ-taut (for the Čech or sheaf cohomology; again on paracompact spaces they agree), i.e.

$$\check{H}^*_{\varphi \cap X}(X) \cong \operatorname*{colim}_{U} \check{H}^*_{\varphi \cap U}(U)$$

with U running over a fundamental system of neighbourhoods. Restricting to the cofinal system of open neighbourhoods one can replace $\check{H}^*_{\varphi \cap U}(U)$ by $H^*_{\varphi \cap U}(U)$. Here one uses the fact that $\varphi \cap U$ is again paracompactifying and that U is homologically locally connected in all dimensions (see Section C.2). Hence the canonical map $\check{H}^*_{\varphi \cap U}(U) \to H^*_{\varphi \cap U}(U)$ is an isomorphism.

Taking a colimit over all closed neighborhoods U of X we get the map from the statement. Hence the proof is reduced to the commutativity of the diagram

$$
\begin{array}{ccc}
H_*(B \cap U, (B \cap U) - X) & \xrightarrow{\ \ g_* \ \ } & H_*(U, U - X) \\
{\scriptstyle -\cap\xi_B}\big\uparrow{\scriptstyle\cong} & {\scriptstyle -\cap g_*\xi_B}\nearrow & {\scriptstyle\cong}\big\uparrow{\scriptstyle -\cap\xi_E} \\
H_c^{m-*}(B \cap U) \xleftarrow[\ g^* \ ]{} & H_{p^{-1}c}^{m-*}(U) \xrightarrow[-\cup h^*\tau]{} & H_c^{n-*}(U)
\end{array}
$$

where $g : B \cap U \to U$ and $h : U \to E$ are the inclusions. The part of the diagram on the left commutes by the naturality of cap products. The commutativity of the triangle on the right is proved using

$$(x \cup h^*\tau) \cap \xi_E = x \cap (h^*\tau \cap \xi_E) = x \cap g_*\xi_B$$

The second equality is an easy consequence of $\tau \cap o_E = i_* o_B$, which was observed just before the proposition. $\qquad\square$

If k^* is an isomorphism this gives an intrinsic description of the transfer map

$$i^! = P_E^{-1} i_* P_B : \check{H}_c^{m-*}(B \cap X) \to \check{H}_c^{n-*}(X)$$

where P_B and P_E are the two duality maps in the statement of the proposition.

In the general case let $D_\varepsilon E$ be the ε-disk subbundle of E, $iD_\varepsilon E$ its interior and $S_\varepsilon E$ its boundary, the ε-sphere bundle. For simplicity we use

$$X_A = A \cap X$$

whenever A is a subspace of E. The proposition still holds with E replaced by $iD_\varepsilon E$ and we have a diagram

$$
\begin{array}{ccc}
H_*(B, B - X) & \xrightarrow{\hspace{5cm}} & H_*(iD_\varepsilon E, iD_\varepsilon E - X) \\
{\scriptstyle -\cap o_B}\big\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\uparrow{\scriptstyle -\cap o_{iD_\varepsilon E}} \\
\check{H}_c^{m-*}(X_B) \xleftarrow{\ k_\varepsilon^* \ } & \check{H}_{p^{-1}c}^{m-*}(X_{iD_\varepsilon E}) \xrightarrow{\ -\cup\tau_\varepsilon' \ } & \check{H}_c^{n-*}(X_{iD_\varepsilon E}) \\
& {\scriptstyle l_\varepsilon^*}\searrow \qquad \big\downarrow & \big\downarrow{\scriptstyle\cong} \\
& \check{H}_c^{m-*}(X_{D_\varepsilon E}) \xrightarrow{\ -\cup\tau_\varepsilon \ } & \check{H}_c^{n-*}(X_{D_\varepsilon E}, X_{S_\varepsilon E})
\end{array}
$$

with τ_ε' and τ_ε being the restrictions of the Thom class

$$\tau \in H_{c_p}^{n-m}(E) \cong H^{n-m}(D_\varepsilon E, S_\varepsilon E)$$

LEMMA D.2. *The following diagram commutes*

$$
\begin{array}{ccc}
H_*(iD_\varepsilon E, iD_\varepsilon E - X) & \xrightarrow{\hspace{5cm}} & H_*(E, E - X) \\
{\scriptstyle -\cap o_{iD_\varepsilon E}}\big\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\uparrow{\scriptstyle -\cap o_E} \\
\check{H}_c^{n-*}(X_{iD_\varepsilon E}) & \xrightarrow{\hspace{5cm}} & \check{H}_c^{n-*}(X) \\
{\scriptstyle\cong}\big\downarrow & \nearrow & \\
\check{H}_c^{n-*}(X_{D_\varepsilon E}, X_{S_\varepsilon E}) & \xleftarrow[\ \cong \ ]{} & \check{H}_c^{n-*}(X, X - (iD_\varepsilon E))
\end{array}
$$

where all the unlabeled maps are induced by respective inclusions.

Proof. The commutativity of the top square is obtained from the commutativity of the outer square of

$$
\begin{array}{ccc}
H_*(U_{iD_\varepsilon E}, U_{iD_\varepsilon E} - X) & \xrightarrow{\iota_*} & H_*(U, U - X) \\
{\scriptstyle -\cap\xi_{iD_\varepsilon E}}\Big\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\uparrow{\scriptstyle -\cap\xi_E} \\
H_c^{n-*}(U_{iD_\varepsilon E}) & \dashrightarrow & H_c^{n-*}(U) \\
\Big\uparrow & & \Big\uparrow \\
H^{n-*}(U_{iD_\varepsilon E}, U_{iD_\varepsilon E} - F) & \xleftarrow[\iota^*]{\cong} & H^{n-*}(U, U - F)
\end{array}
$$

(note that $\xi_E = \iota_*(\xi_{iD_\varepsilon E})$) first by passing to the colimit over all compact $F \subseteq U_{iD_\varepsilon E}$ to get commutativity of the upper square and then by passing to the colimit over all closed neighbourhoods U of X. The bottom part of the diagram from the statement commutes obviously. $\qquad\square$

By taking a (co)limit $\varepsilon \to 0$ both the maps k_ε^* and l_ε^* induce isomorphisms

$$
\begin{array}{ccc}
\check{H}_c^{m-*}(X_B) & \xleftarrow[\cong]{\{k_\varepsilon^*\}} & \mathrm{colim}\,\check{H}_{p^{-1}c}^{m-*}(X_{iD_\varepsilon E}) \\
 & {\scriptstyle \{l_\varepsilon^*\}}\ \ {\scriptstyle \cong}\nwarrow & \Big\downarrow{\scriptstyle \cong} \\
 & & \mathrm{colim}\,\check{H}_c^{m-*}(X_{D_\varepsilon E})
\end{array}
$$

and we get at least some (intrinsic) description of the transfer map by pasting the two diagrams together.

Theorem D.3. *The transfer map* $i^! : \check{H}_c^{m-*}(B \cap X) \to \check{H}_c^{n-*}(X)$ *is the colimit of*

$$
\check{H}_c^{m-*}(X_B) \xleftarrow{l_\varepsilon^*} \check{H}_c^{m-*}(X_{D_\varepsilon E}) \xrightarrow{-\cup\tau_\varepsilon} \check{H}_c^{n-*}(X_{D_\varepsilon E}, X_{S_\varepsilon E}) \xleftarrow{\cong}
$$

$$
\xleftarrow{\cong} \check{H}_c^{n-*}(X, X - (iD_\varepsilon E)) \longrightarrow \check{H}_c^{n-*}(X)
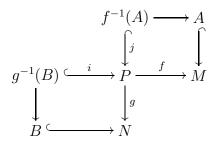$$

where the first map becomes an isomorphism in the colimit. $\qquad\square$

APPENDIX E

Transversality of maps from preimages of other maps

This appendix is rather unrelated to the rest of the thesis. It seemed to be important for the proof of the main theorem but turned out not to be. It gives an equivalent condition to a transversality of a restriction of a fixed map $g : P \to M$ to a preimage $f^{-1}(A)$ of a submanifold along a map $f : P \to N$ in terms of transversality conditions of f itself. An the end we prove that this property is generic in the sense that such maps form a residual subset.

LEMMA E.1. *Let*

$$
\begin{array}{ccc}
f^{-1}(A) & \longrightarrow & A \\
\downarrow{\scriptstyle j} & & \uparrow \\
g^{-1}(B) \overset{i}{\hookrightarrow} P & \overset{f}{\longrightarrow} & M \\
\downarrow & & \downarrow{\scriptstyle g} \\
B \hookrightarrow & N &
\end{array}
$$

be a diagram of smooth manifolds and smooth maps between them where the maps denoted by $\hookrightarrow$ are inclusions of submanifolds and where we assume that $f \pitchfork A$ and $g \pitchfork B$. Then the following conditions are equivalent:

(i) $gj \pitchfork B$
(ii) $fi \pitchfork A$
(iii) $f^{-1}(A) \pitchfork g^{-1}(B)$
(iv) $(f,g) \pitchfork A \times B$ *where* $(f,g) : P \to M \times N$

PROOF. Let us start first by showing that (i), (ii) and (iii) are equivalent. Because (iii) is symmetric it is enough to show the equivalence of (i) and (iii). But (i) is equivalent to the map

$$T_x f^{-1}(A) \longrightarrow T_{g(x)}N/T_{g(x)}B$$

induced by g being surjective for every $x \in f^{-1}(A) \cap g^{-1}(B)$. Because of the assumption $g \pitchfork B$, we have a commutative diagram

$$
\begin{array}{ccc}
T_x f^{-1}(A) & \longrightarrow & T_{g(x)}N/T_{g(x)}B \\
\downarrow & \nearrow {\scriptstyle \cong} & \\
T_x P/T_x g^{-1}(B) & &
\end{array}
$$

and so (i) is equivalent to (iii).

Let us deal with (iv) now (which we actually do not need in the following). It can be phrased as a surjectivity of the map (induced by (f, g))

$$\varphi : T_x P \longrightarrow (T(M \times N)/T(A \times B))_{(f(x),g(x))} \cong (TM/TA)_{f(x)} \oplus (TN/TB)_{g(x)}$$

for every $x \in f^{-1}(A) \cap g^{-1}(B)$. Assuming (i) and (ii) the image under φ of $T_x f^{-1}(A)$ is $0 \oplus (TN/TB)_{g(x)}$ and the image of $T_x g^{-1}(B)$ is $(TM/TA)_{f(x)} \oplus 0$ and so φ is surjective. Assuming the surjectivity of φ on the other hand we can find $u \in T_x P$ mapping to $(0, v)$ by φ for any choice of v. Necessarily $u \in T_x f^{-1}(A)$ and so the map

$$T_x f^{-1}(A) \longrightarrow T_{g(x)} N / T_{g(x)} B$$

is surjective. This is condition (i). $\qquad\square$

CONSTRUCTION E.2. Let D be a d-dimensional manifold and

$$\mathrm{Diff}(\mathbb{R}^d, 0) = G_{0,\mathrm{diff}}(\mathbb{R}^d, \mathbb{R}^d)_0$$

the group[1] of germs at 0 of local diffeomorphisms $\mathbb{R}^d \to \mathbb{R}^d$ fixing 0. Define a principal $\mathrm{Diff}(\mathbb{R}^d, 0)$-bundle

$$\mathrm{Charts}_D = G_{0,\mathrm{diff}}(\mathbb{R}^d, D) \xrightarrow{\mathrm{ev}_0} D$$

of germs at 0 of local diffeomorphisms $\mathbb{R}^d \to D$. If F is any manifold with a (smooth in some sense) action of $\mathrm{Diff}(\mathbb{R}^d, 0)$ then we can construct an associated bundle

$$D[F] := \mathrm{Charts}_D \times_{\mathrm{Diff}(\mathbb{R}^d, 0)} F \longrightarrow D$$

Any bundle of this form will be called *local*. Observe that this construction is functorial in D on the category of d-dimensional manifolds and local diffeomorphisms. As an example the bundle $J^r(D, N)$ of r-jets of maps $D \to N$ is a local bundle as

$$D[J_0^r(\mathbb{R}^d, N)] = \mathrm{Charts}_D \times_{\mathrm{Diff}(\mathbb{R}^d, 0)} J_0^r(\mathbb{R}^d, N) \cong J^r(D, N) \tag{E.1}$$

where $J_0^r(\mathbb{R}^d, N)$ is the subspace of $J^r(\mathbb{R}^d, N)$ of r-jets with source 0. The bijection is provided by the map

$$[u, \alpha] \mapsto \alpha \circ j_{u(0)}^r(u^{-1})$$

Having a $\mathrm{Diff}(\mathbb{R}^d, 0)$-invariant submanifold $B \subset J_0^r(\mathbb{R}^d, N)$ we get an associated sub-bundle $D[B] \subset J^r(D, N)$ for any d-dimensional manifold D. This allows us to talk about jet transversality conditions on a map $D \to N$ without specifying what D (and hence also $J^r(D, N)$) is. Let us fix such a $B \subset J_0^r(\mathbb{R}^d, N)$.

Using (E.1) one can easily see that $j_x^r(h) \in J_x^r(D, N)$ lies in $D[B]$ iff in some (and hence any) chart $u \in G_{0,\mathrm{diff}}(\mathbb{R}^d, D)_x$ the local expression $j_0^r(hu)$ of $j_x^r(h)$ in the chart u lies in B.

LEMMA E.3. *For $h : D \to N$ the following conditions are equivalent*

 (i) $h_* : J_{0,\mathrm{imm}}^r(\mathbb{R}^d, D) \to J_0^r(\mathbb{R}^d, N)$ *is transverse to B*
 (ii) $j^r(h) : D \to J^r(D, N)$ *is transverse to $D[B]$*

where the "imm" index means we take the subspace of jets of immersions.

---

[1]If we wanted to give $\mathrm{Diff}(\mathbb{R}^d, 0)$ a topology we could do so by inducing the topology via the map $\mathrm{Diff}(\mathbb{R}^d, 0) \to J_0^\infty(\mathbb{R}^d, \mathbb{R}^d)_0$. Or one could replace $\mathrm{Diff}(\mathbb{R}^d, 0)$ by its image - the subspace of invertible ∞-jets.

Proof. Taking the associated bundles (i) is obviously equivalent to the transversality of

$$h_* : J^r_{\mathrm{imm}}(D, D) \to J^r(D, N)$$

to $D[B]$. Let $j^r_x(k) \in J^r_{x,\mathrm{imm}}(D, D)_y$ be an r-jet of a diffeomorphism $k : V \to W$ between open subsets of D. Then we have a diagram

$$
\begin{array}{ccccc}
J^r_{\mathrm{imm}}(V, D) & \xrightarrow{h_*} & J^r(V, N) & \longleftarrow & V[B] \\
\cong \downarrow{k_*} & & \cong \downarrow{k_*} & & \cong \downarrow{k_*} \\
J^r_{\mathrm{imm}}(W, D) & \xrightarrow{h_*} & J^r(W, N) & \longleftarrow & W[B]
\end{array}
$$

Now $j^r_x(k)$ in the top left corner is mapped by k_* down to $j^r_y(\mathrm{id})$. Hence we see that it is enough (equivalent) to check the transversality only at $j^r_y(\mathrm{id})$'s for all $y \in D$ for which $h_*(j^r_y(\mathrm{id})) = j^r_y(h) \in D[B]$. For such y the same diagram shows that every $j^r_x(k)$ with target y is mapped by h_* to $D[B]$. Thus the whole fibre over y of the target map

$$J^r_{\mathrm{imm}}(D, D) \xrightarrow{\tau} D$$

is mapped to $D[B]$. The target map τ has a section

$$j^r(\mathrm{id}) : D \to J^r_{\mathrm{imm}}(D, D)$$

and so (i) is finally equivalent to the composite

$$D \xrightarrow{j^r(\mathrm{id})} J^r_{\mathrm{imm}}(D, D) \xrightarrow{h_*} J^r(D, N)$$

being transverse to $D[B]$. This is (ii).   □

We say that a map $g : P \to N$ is transverse to B, denoted $g \pitchfork B$, if

$$g_* : J^r_{0,\mathrm{imm}}(\mathbb{R}^d, P) \to J^r_0(\mathbb{R}^d, N)$$

is transverse to B. This is the case for example if g is a submersion. When $r = 0$ this is equivalent to the usual transversality of a map to a submanifold. Let $f \pitchfork A$ where $f : P \to M$ and $i : A \subset M$ is a submanifold. Then we have the following diagram

$$
\begin{array}{ccccc}
J^r_{0,\mathrm{imm}}(\mathbb{R}^d, f^{-1}(A)) & \lhook\joinrel\longrightarrow & J^r_0(\mathbb{R}^d, f^{-1}(A)) & \longrightarrow & J^r_0(\mathbb{R}^d, A) \\
\downarrow{j_*} & & \downarrow{j_*} & & \downarrow{i_*} \\
J^r_{0,\mathrm{imm}}(\mathbb{R}^d, P) & \lhook\joinrel\longrightarrow & J^r_0(\mathbb{R}^d, P) & \xrightarrow{f_*} & J^r_0(\mathbb{R}^d, M)
\end{array}
$$

where both squares are pullbacks. This can be easily seen in local coordinates. Also i_* is a submanifold inclusion and $f_* \pitchfork i_*$. Combining Lemma E.1 with Lemma E.3 we get:

Lemma E.4. *Let*

$$
\begin{array}{ccc}
f^{-1}(A) & \longrightarrow & A \\
\downarrow{j} & & \uparrow \\
P & \xrightarrow{f} & M \\
\downarrow{g} & & \\
N & &
\end{array}
$$

be a diagram of smooth manifolds and smooth maps between them where the maps denoted by $\hookrightarrow$ *are inclusions of submanifolds. Let us assume that* $f \pitchfork A$ *and* $g \pitchfork B$ *where* $B \subset J_0^r(\mathbb{R}^d, N)$ *is a* $\mathrm{Diff}(\mathbb{R}^d, 0)$-*invariant submanifold with* $d = \dim P + \dim A - \dim M$. *Then the following conditions are equivalent:*

(i) $j^r(gj) \pitchfork f^{-1}(A)[B]$, *where*

$$j^r(gj) : f^{-1}(A) \to J^r(f^{-1}(A), N)$$

is the jet prolongation

(ii) $f_*|_Y \pitchfork J_0^r(\mathbb{R}^d, A)$, *where* $Y = (g_*)^{-1}(B)$ *is defined by a pullback diagram*

$$\begin{array}{ccc} Y & \hookrightarrow & J_{0,\mathrm{imm}}^r(\mathbb{R}^d, P) \\ \downarrow & & \downarrow g_* \\ B & \hookrightarrow & J_0^r(\mathbb{R}^d, N) \end{array}$$

PROOF. Applying Lemma E.1 to the diagram

$$\begin{array}{ccc} J_{0,\mathrm{imm}}^r(\mathbb{R}^d, f^{-1}(A)) & \longrightarrow & J_0^r(\mathbb{R}^d, A) \\ \downarrow j_* & & \downarrow i_* \\ Y \hookrightarrow J_{0,\mathrm{imm}}^r(\mathbb{R}^d, P) & \xrightarrow{f_*} & J_0^r(\mathbb{R}^d, M) \\ \downarrow & & \downarrow g_* \\ B \hookrightarrow & & J_0^r(\mathbb{R}^d, N) \end{array}$$

gives an equivalence of (ii) with transversality of

$$(gj)_* : J_{0,\mathrm{imm}}^r(\mathbb{R}^d, f^{-1}(A)) \longrightarrow J_0^r(\mathbb{R}^d, N)$$

to B. By Lemma E.3 this is equivalent to (i). □

Now that we know how f controls the transversality of a map defined on the preimage $f^{-1}(A)$ of some submanifold, we would like to see that this transversality condition (any of the two equivalent conditions in Lemma E.4) is generic. This is indeed the case. We first prove a more general result which at the same time happens to generalize the Thom Transversality Theorem.

LEMMA E.5. *Let* D, M, N *be manifolds,* $Y \subset J_{\mathrm{imm}}^r(D, M)$ *and* $Z \subset J^r(D, N)$ *submanifolds. Let us further assume that* $\sigma_Y \pitchfork \sigma_Z$, *where*

$$\sigma_Y = \sigma|_Y : Y \subset J^r(D, M) \to D \quad \text{and} \quad \sigma_Z = \sigma|_Z : Z \subset J^r(D, N) \to D$$

are the restrictions of the source maps. For a smooth map $f : M \to N$ *let* $f_*|_Y$ *denote the map*

$$Y \subset J_{\mathrm{imm}}^r(D, M) \xrightarrow{f_*} J^r(D, N)$$

Then the set

$$\mathfrak{X} := \{ f \in C^\infty(M, N) \mid f_*|_Y \pitchfork Z \}$$

is residual in $C^\infty(M, N)$ *with the strong topology, and open if* Z *is closed (as a subset) and* $\tau_Y : Y \to M$ *proper.*

Proof. This is an application of Lemma 3.1. We have a map

$$\alpha : C^\infty(M, N) \to C^\infty(Y, J^r(D, N))$$

sending f to $f_*|_Y$. This map is continuous in the weak topologies. Clearly $\mathfrak{X} = \{f \in C^\infty(M, N) \mid \alpha(f) \pitchfork Z\}$. We have to verify the conditions of Lemma 3.1.

Let $f_0 \in C^\infty(M, N)$ and $K \subseteq Y$, $L \subseteq Z$ compact disks. We can assume that $\tau(K)$ lies in a coordinate chart $\mathbb{R}^m \cong U \subseteq M$ and that $\tau(L)$ lies in a coordinate chart $\mathbb{R}^n \cong V \subseteq N$. We use these charts to identify U with $\mathbb{R}^m$ and V with $\mathbb{R}^n$ when needed. Let $\lambda : U \to \mathbb{R}$ be a compactly supported function such that $\lambda = 1$ on a neighborhood of $\tau(K) \cap f_0^{-1}(\tau(L))$ and such that $\lambda = 0$ on $U - f_0^{-1}(V)$.

We set $P := J^r_*(\mathbb{R}^m, \mathbb{R}^n) \cong J^r_*(U, V)$ and identify it with the space of polynomial mappings $U \to V$. Then we get a map

$$\beta : P \to C^\infty(M, N)$$

sending g to the function $f_0 + \lambda g$ where the operations are interpreted inside V via the chart. It is continuous in the strong topology and the map

$$\gamma = (\alpha\beta)^\sharp : P \times Y \to J^r(D, N)$$

is smooth. Thus it is enough to show that (after a suitable restriction) $\gamma \pitchfork Z$. Clearly γ sends $(g, j^r_x(h))$ to $j^r_x((f_0 + \lambda g)h)$. Suppose now that $h(x) \in W := \mathrm{int}\,\lambda^{-1}(1)$ so that this equals to $j^r_x(f_0 h + gh)$. By restriction we get a map

$$\delta : P \cong P \times \{j^r_x(h)\} \xrightarrow{\gamma} J^r_x(D, V)$$

In the affine structure on $J^r_x(D, V)$ inherited from the chart δ is clearly affine. Identifying P with $J^r_{h(x)}(U, V)$ the linear part of δ is just a precomposition with h

$$h^* : J^r_{h(x)}(U, V) \to J^r_x(D, V)$$

The map h, being an immersion, has (locally - near x) a left inverse π which then gives a right inverse π^* of h^* and so the linear part of δ is surjective and hence it is a submersion.

In the horizontal direction our transversality condition $\sigma_Y \pitchfork \sigma_Z$ applies and so $\gamma \pitchfork Z$ on $P \times \tau_Y^{-1}(W)$. If $f : M \to N$ is close enough to f_0 then

$$\tau(K) \cap f^{-1}(\tau(L)) \subseteq W$$

and in particular there is a neighbourhood P' of 0 in P such that $\beta(P')$ consists only of such maps. Therefore the restriction of γ to

$$P' \times K \longrightarrow J^r(D, N)$$

is transverse to Z.

If τ_Y happens to be proper then α is continuous even in strong topologies and $\mathfrak{X}$ is a preimage of the open subset of maps $f : Y \to J^r(D, N)$ transverse to Z. $\qquad\square$

Corollary E.6 (Thom Transversality Theorem). *Let M, N be manifolds, $Z \subseteq J^r(M, N)$ a submanifold. Then the set*

$$\mathfrak{X} := \{f \in C^\infty(M, N) \mid j^r f \pitchfork Z\}$$

is residual in $C^\infty(M, N)$. If Z is closed (as a subset) then it is also open.

PROOF. We apply Lemma E.5 to $D = M$ and

$$Y = M \xhookrightarrow{\ \ j^r(\mathrm{id})\ \ } J^r(M, M)$$

with maps id and σ to M.

As $\sigma_Y = \mathrm{id} = \tau_Y$ it is both proper and transverse to σ_Z for any Z. □

COROLLARY E.7. *Let* M, N *be manifolds,* $Y \subset J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M)$ *and* $Z \subset J^r_0(\mathbb{R}^d, N)$ *submanifolds. Then the set*

$$\left\{ f \in C^\infty(M, N) \mid f_*|_Y \pitchfork Z \right\}$$

is residual in $C^\infty(M, N)$ *with the strong topology, and open if* Z *is closed (as a subset) and* $\tau_Y : Y \to M$ *proper.*

PROOF. Under the natural identifications
$$\mathbb{R}^d \times J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M) \cong J^r_{\mathrm{imm}}(\mathbb{R}^d, M)$$
$$\mathbb{R}^d \times J^r_0(\mathbb{R}^d, N) \cong J^r(\mathbb{R}^d, N)$$

we can apply Lemma E.5 to $D = \mathbb{R}^d$, M, N and
$$0 \times Y \subseteq J^r_{\mathrm{imm}}(\mathbb{R}^d, M)$$
$$\mathbb{R}^d \times Z \subseteq J^r(\mathbb{R}^d, N)$$

□

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